On spectrum of a periodic operator with a small localized perturbation

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Abstract

We study the spectrum of a periodic self-adjoint operator on the axis perturbed by a small localized nonself-adjoint operator. It is shown that the continuous spectrum is independent of the perturbation, the residual spectrum is empty, and the point spectrum has no finite accumulation points. We address the existence of the embedded eigenvalues. We establish the necessary and sufficient conditions of the existence of the eigenvalues and construct their asymptotics expansions. The asymptotics expansions for the associated eigenfunctions are also obtained. The examples are given.

1 Introduction

It is well-known that the spectrum of a self-adjoint periodic one-dimensional differential operator consists of zones separated by lacunas (see, for instance, ([1, Ch. V, Sec. 56], [2, Ch. 5]). Perturbation of such operator by a rapidly decreasing potential does not change the continuous part of the spectrum but produces the isolated eigenvalues in the lacunas. The existence and the number of such eigenvalues were studied, for instance, in [3], [4], [5], [6]. It was shown that the number of the eigenvalues in each lacuna is finite and there are at most two eigenvalues in the distant lacunas. In [6] they also studied the case when the perturbing potential is multiplied by a small coupling constant. It was established that each lacuna contains at most two eigenvalues. The necessary and sufficient conditions exactly determining the number of the eigenvalues in a given lacuna were adduced.

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In the present paper we study the spectrum of a self-adjoint periodic second-order differential operator on the axis perturbed by a linear operator of the form $\varepsilon\mathcal{L}_\varepsilon$, where ε is a small positive parameter. The main feature of $\mathcal{L}_\varepsilon$ is that it is localized in the following sense. The support of the function $\mathcal{L}_\varepsilon u$ lies inside some fixed finite segment $\overline{Q}$ and this function is fully determined by the values the argument u takes on Q.

The main difference of the perturbation we study from the cases in the papers cited is that we do not assume the self-adjointness neither for $\mathcal{L}_\varepsilon$ nor for the perturbed operator. Moreover, the set of possible perturbation described by such perturbation includes, apart from the potentials, a wide class of examples of various nature like differential operator, integral operator, linear functional (see last section).

In the paper we show that the continuous spectrum of the perturbed operator coincides with the spectrum of the unperturbed one. We also establish that the residual spectrum is empty, while the point spectrum consists of at most countably many eigenvalues of finite multiplicity and has no finite accumulation points. We also give an example of the perturbation which originates an embedded eigenvalue. We notice that similar phenomenon could not rise in the problems considered in [3], [4], [5], [6]. We also provide the sufficient conditions guaranteeing absence of the embedded eigenvalues. It is established that the perturbed eigenvalues tend either to infinity or to the edges of non-degenerated lacunas in the continuous spectrum. We prove that there exists at most one such eigenvalue in the vicinity of an edge of a given non-degenerated lacuna. We give the criteria for the existence of this eigenvalue and construct its asymptotics expansions as well as the expansion for the associated eigenfunction.

In conclusion let us describe briefly the structure of the paper. In the following section we formulate the problem and present the main results. In the third section we prove the general theorem on the position of the perturbed spectrum and show that continuous spectrum is independent of the perturbation and the residual one is empty. In the fourth section we show the absence of the embedded eigenvalues in the finite part of spectrum if ε is small enough. The fifth section is devoted to the countability, convergence and some other properties of the point spectrum. In the sixth section we study the existence of the embedded eigenvalues. In the seventh section certain auxiliary statements are proven. These statements are employed in the eighth section where we construct the asymptotics for the eigenvalues converging to the edges of the non-degenerate lacunas in the continuous spectrum. In the last ninth section we give examples of the perturbation and apply to them the general results of the work.

2 Formulation of the problem and the main results

Let

$$\mathcal{H}_0 := -\frac{d}{dx}p\frac{d}{dx} + q$$

be a self-adjoint operator in $L_2(\mathbb{R})$ with the domain $W_2^2(\mathbb{R})$. Here $p = p(x)$ is 1-periodic piecewise continuously differentiable real function, $q = q(x)$ is 1-periodic piecewise continuous real function, and

$$p(x) \geqslant p_0 > 0, \quad x \in \mathbb{R}. \tag{2.1}$$

Without loss of generality throughout the paper we assume that $p(0) = 1$.

Let $\mathcal{L}_\varepsilon : W_2^2(Q) \to L_2(Q)$ be a linear operator bounded uniformly in ε, and generally speaking unbounded as an operator in $L_2(Q)$. We introduce the operator mapping $W_{2,loc}^2(\mathbb{R})$ into $L_2(\mathbb{R})$ by the following rule: an element from $W_{2,loc}^2(\mathbb{R})$ is restricted to Q, then the operator $\mathcal{L}_\varepsilon$ is applied, and the result is continued by zero outside Q. Such operator is naturally to indicate by $\mathcal{L}_\varepsilon$. Clearly, this is an unbounded operator in $L_2(\mathbb{R})$ with the domain $W_{2,loc}^2(\mathbb{R})$.

We indicate $\mathcal{H}_\varepsilon := (\mathcal{H}_0 - \varepsilon\mathcal{L}_\varepsilon)$ considering it as an operator in $L_2(\mathbb{R})$ having $W_2^2(\mathbb{R})$ as the domain. The operator $\mathcal{H}_\varepsilon$ is closed (see Lemma 3.2).

The main aim of this paper is to study the behaviour of the spectrum of the operator $\mathcal{H}_\varepsilon$ as $\varepsilon \to 0$. Before presenting the main results we introduce additional notations and remind some known facts.

We will employ the symbols $\sigma(\cdot)$, $\sigma_{\mathrm{c}}(\cdot)$ and $\sigma_{\mathrm{p}}(\cdot)$ to indicate the spectrum, continuous spectrum, and the point spectrum, while

$$\sigma_{\mathrm{r}}(\cdot) := \sigma(\cdot)\setminus(\sigma_{\mathrm{c}}(\cdot) \cup \sigma_{\mathrm{p}}(\cdot))$$

is the residual spectrum. It is known [2, Ch. 2, Sec. 2.2, 2.3, Ch. 5, Sec. 5.3] that the operator $\mathcal{H}_0$ has a band spectrum

$$\sigma(\mathcal{H}_0) = \sigma_{\mathrm{c}}(\mathcal{H}_0) = \bigcup_{n=0}^{\infty} [\mu_n^+, \mu_{n+1}^-], \tag{2.2}$$

where the numbers

$$\mu_0^+ < \mu_1^- \leqslant \mu_1^+ < \mu_2^- \leqslant \mu_2^+ < \mu_3^- \leqslant \mu_3^+ < \ldots$$

are simple eigenvalues of the boundary value problems

$$\begin{gathered}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right)\phi_n^\pm = \mu_n^\pm\phi_n^\pm, \quad x \in (0,1),\\ \phi_n^\pm(0) + (-1)^{n+1}\phi_n^\pm(1) = 0, \quad \frac{d\phi_n^\pm}{dx}(0) + (-1)^{n+1}\frac{d\phi_n^\pm}{dx}(1) = 0.\end{gathered} \tag{2.3}$$

For $a \in \mathbb{C}$, $\delta > 0$ we denote $S_\delta(a) := \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| < \delta\}$.

We are ready to formulate the main results of the paper.

Theorem 2.1. *There exist positive $\delta_i = \delta_i(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$, $i = 1, 2$, so that for ε small enough the inclusion $\sigma(\mathcal{H}_\varepsilon) \subset S_{\delta_2(\varepsilon)}\big(\mu_0^+ - \delta_1(\varepsilon)\big)$ holds true.*

Theorem 2.2. *For ε small enough the identities $\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon) = \sigma_{\mathrm{c}}(\mathcal{H}_0)$, $\sigma_{\mathrm{r}}(\mathcal{H}_\varepsilon) = \emptyset$ are valid.*

Theorem 2.3. *The point spectrum of the operator $\mathcal{H}_\varepsilon$ consists of countably many eigenvalues of finite multiplicities and has no finite accumulation points.*

Theorem 2.4. *Let K be a compact set in the complex plane such that $K \cap \sigma_{\mathrm{c}}(\mathcal{H}_0) \neq \emptyset$. Then for ε small enough the set $\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon) \cap K$ contains no embedded eigenvalues.*

We stress that this theorem does not exclude the presence of the embedded eigenvalues tending to infinity as $\varepsilon \to 0$. In the sixth section we will give an example of the operator $\mathcal{H}_\varepsilon$ which has an embedded eigenvalue. In the following theorem we provide the sufficient conditions of the absence of such eigenvalues.

Theorem 2.5. *Assume that at least one of the following conditions is valid*

(1). For any subinterval $\widetilde{Q} \subseteq Q$ the estimate

$$\|\mathcal{L}_\varepsilon u\|_{L_2(\widetilde{Q})} \leqslant C \|u\|_{W_2^2(\widetilde{Q})} \tag{2.4}$$

holds true with the constant C independent of ε and $\widetilde{Q}$.

(2). The operator $\mathcal{L}_\varepsilon$ can be represented as

$$\mathcal{L}_\varepsilon = \frac{d}{dx} a_\varepsilon \frac{d}{dx} + \widetilde{\mathcal{L}}_\varepsilon,$$

where a_ε is piecewise continuously differentiable function having support inside $\overline{Q}$ and satisfying the relation

$$\varepsilon \max_{\overline{Q}} |a'_\varepsilon(x)| \xrightarrow[\varepsilon\to 0]{} 0, \tag{2.5}$$

and $\widetilde{\mathcal{L}}_\varepsilon : W_2^1(Q) \to L_2(Q)$ is a linear operator bounded uniformly in ε.

Then for ε small enough the continuous spectrum of $\mathcal{H}_\varepsilon$ contains no embedded eigenvalues.

Theorem 2.6. *Let K be a compact set in the complex plane such that $K \cap \sigma_{\mathrm{c}}(\mathcal{H}_0) \neq \emptyset$. Then for ε small enough each of the eigenvalues $\mathcal{H}_\varepsilon$ not leaving K for all ε small enough converges in the limit $\varepsilon \to 0$ to one of the edges of a non-degenerate lacuna in the part of the spectrum of $\mathcal{H}_0$ lying inside K.*

Let $\theta_i(x,\lambda)$ be the solutions to the equation

$$\left(-\frac{d}{dx}p\frac{d}{dx}+q-\lambda\right)v=0,\quad x\in\mathbb{R}, \tag{2.6}$$

satisfying the initial conditions

$$\theta_1(0,\lambda)=1,\quad \frac{d\theta_1}{dx}(0,\lambda)=0,\qquad \theta_2(0,\lambda)=0,\quad \frac{d\theta_2}{dx}(0,\lambda)=1, \tag{2.7}$$

where λ is a complex parameter. For the sake of brevity hereinafter we denote $\theta_i(\lambda):=\theta_i(1,\lambda)$, $\theta_i'(\lambda):=\frac{d\theta_i}{dx}(1,\lambda)$, $i=1,2$. We set $D(\lambda):=\theta_1(\lambda)+\theta_2'(\lambda)$.

Let $\mu_n^\pm$ be one of the edges of a non-degenerate lacuna in the spectrum of $\mathcal{H}_0$. We choose the eigenfunctions of the problem (2.3) being real and continue them 1-periodically over the axis for even n and 1-antiperiodically for odd n. It is clear that the functions continued are twice piecewise continuously differentiable. We normalize them as follows

$$|\phi_n^\pm(0)|^2+\left|\frac{d\phi_n^\pm}{dx}(0)\right|^2=|\theta_1'(\mu_n^\pm)|+|\theta_2(\mu_n^\pm)|. \tag{2.8}$$

We will show below that the right hand side of this identity is non-zero (see Item 1 of Lemma 5.3), and this is why this normalization makes sense.

Let $\mathcal{G}_{n,0}^\pm$ be an integral operator defined on $L_2(Q)$:

$$(\mathcal{G}_{n,0}^\pm f)(x):=\int\limits_{\mathbb{R}} G_{n,0}^\pm(x,t)f(t)\,\mathrm{d}t, \tag{2.9}$$

$$G_{n,0}^\pm(x,t):=\frac{1}{2}\begin{cases}\theta_1(t,\mu_n^\pm)\theta_2(x,\mu_n^\pm)-\theta_1(x,\mu_n^\pm)\theta_2(t,\mu_n^\pm), & t>x,\\ \theta_1(x,\mu_n^\pm)\theta_2(t,\mu_n^\pm)-\theta_1(t,\mu_n^\pm)\theta_2(x,\mu_n^\pm), & t<x.\end{cases}$$

Since $\mathcal{G}_{n,0}^\pm : L_2(Q)\to W_2^2(Q)$ is a bounded operator, it follows that for ε small enough the operator $\mathcal{L}_\varepsilon\mathcal{G}_{n,0}^\pm : L_2(Q)\to L_2(Q)$ is bounded uniformly in ε, and thus for ε small enough the bounded operator

$$\mathcal{A}_n^\pm(\varepsilon,0):=\left(\mathrm{I}-\varepsilon\mathcal{L}_\varepsilon\mathcal{G}_{n,0}^\pm\right)^{-1}.$$

is well-defined in $L_2(Q)$. Hereinafter I is the identity mapping. By the dot we will indicate the differentiation w.r.t. λ.

Theorem 2.7. *Let $\mu_n^\pm$ be one of the edges of a non-degenerate lacuna in the spectrum of $\mathcal{H}_0$. Then the operator H_ε has at most one eigenvalue $\lambda_{\varepsilon,n}^\pm$ converging to $\mu_n^\pm$ as $\varepsilon\to 0$. This eigenvalue exists if and only if*

$$\pm\operatorname{Re}\left(\phi_n^\pm,\mathcal{A}_n^\pm(\varepsilon,0)\mathcal{L}_\varepsilon\phi_n^\pm\right)_{L_2(Q)}>0. \tag{2.10}$$

If exists, this eigenvalue is simple and has the asymptotics expansion

$$\lambda_{\varepsilon,n}^{\pm} = \mu_n^{\pm} \mp \frac{\varepsilon^2}{4|\dot{D}(\mu_n^{\pm})|}\big(\phi_n^{\pm}, \mathcal{A}_n^{\pm}(\varepsilon,0)\mathcal{L}_\varepsilon\phi_n^{\pm}\big)_{L_2(Q)}^2(1+\mathcal{O}(\varepsilon^2)), \tag{2.11}$$

$$\lambda_{\varepsilon,n}^{\pm} = \mu_n^{\pm} \mp \varepsilon^2\big(k_{n,\varepsilon}^{\pm,1} + \varepsilon k_{n,\varepsilon}^{\pm,2}\big)^2 + \mathcal{O}(\varepsilon^4|k_{n,\varepsilon}^{\pm,1}| + \varepsilon^5), \tag{2.12}$$

$$k_{n,\varepsilon}^{\pm,1} := \pm\frac{\big(\mathcal{L}_\varepsilon\phi_n^{\pm}, \phi_n^{\pm}\big)_{L_2(Q)}}{2\sqrt{|\dot{D}(\mu_n^{\pm})|}}, \quad k_{n,\varepsilon}^{\pm,2} := \pm\frac{\big(\mathcal{L}_\varepsilon\mathcal{G}_{n,0}^{\pm}\mathcal{L}_\varepsilon\phi_n^{\pm}, \phi_n^{\pm}\big)_{L_2(Q)}}{2\sqrt{|\dot{D}(\mu_n^{\pm})|}}. \tag{2.13}$$

The asymptotics expansion for the associated eigenfunction reads as follows

$$\psi_{\varepsilon,n}^{\pm} = \phi_n^{\pm} + \varepsilon\mathcal{G}_{n,0}^{\pm}\mathcal{L}_\varepsilon\phi_n^{\pm} + \mathcal{O}(\varepsilon^2) \tag{2.14}$$

in the norm of $W_2^2(\alpha_1,\alpha_2)$ for any $\alpha_1,\alpha_2 \in \mathbb{R}$.

Remark 2.1. In Lemma 5.3 we will give the formulas for $\dot{D}(\mu_n^{\pm})$, which imply in particular that $\dot{D}(\mu_n^{\pm}) \neq 0$, if $\mu_n^{\pm}$ is an edge of a non-degenerate lacuna.

Theorems 2.6, 2.7 yield immediately

Corollary 2.8. *Let K be a compact set in the complex plane. Then each of the eigenvalues of $\mathcal{H}_\varepsilon$ lying inside K for all ε small enough is simple.*

Theorem 2.9. *Let $\mu_n^{\pm}$ be one of the edges of a non-degenerate lacuna in the spectrum of $\mathcal{H}_0$. If*

$$\operatorname{Re}\big(k_{n,\varepsilon}^{\pm,1} + \varepsilon k_{n,\varepsilon}^{\pm,2}\big) \geqslant C(\varepsilon)\varepsilon^2, \quad C(\varepsilon) \xrightarrow[\varepsilon\to 0]{} +\infty, \tag{2.15}$$

there exists the eigenvalue $\lambda_{\varepsilon,n}^{\pm}$, and the identities (2.11), (2.12) hold true. In the case

$$\operatorname{Re}\big(k_{n,\varepsilon}^{\pm,1} + \varepsilon k_{n,\varepsilon}^{\pm,2}\big) \leqslant -C(\varepsilon)\varepsilon^2, \quad C(\varepsilon) \xrightarrow[\varepsilon\to 0]{} +\infty, \tag{2.16}$$

the operator $\mathcal{H}_\varepsilon$ has no eigenvalues converging to $\mu_n^{\pm}$ as $\varepsilon \to 0$.

Remark 2.2. We also obtain the explicit formula for the eigenfunction $\psi_{\varepsilon,n}^{\pm}$ and describe how it behaves at infinity (see (8.4), (8.5), (8.6)).

Remark 2.3. We notice that in the particular case $p \equiv$ *const*, $q \equiv$ *const* the continuous spectrum of $\mathcal{H}_0$ coincides with the semi-axis $[q,+\infty)$ and has no internal lacunas. In this case the semi-infinite lacuna $(-\infty,q)$ is the only non-degenerate one and Theorems 2.6, 2.7, 2.9 describes the behaviour of the eigenvalues in the vicinity of the point $\mu_0^+ = q$.

3 Proof of Theorems 2.1, 2.2

We denote $B_r(a) := \{\lambda \in \mathbb{C} : |\lambda - a| < r\}$. For any pair of non-empty sets $M_1, M_2 \subset \mathbb{C}$ we indicate

$$\operatorname{dist}(M_1, M_2) := \inf_{\substack{\lambda_1 \in M_1 \\ \lambda_2 \in M_2}} |\lambda_1 - \lambda_2|.$$

Lemma 3.1. *Let M be a non-empty closed set in the complex plane such that $M \cap \sigma(\mathcal{H}_0) = \emptyset$ and for some $a \in \mathbb{R}$, $\delta \in (0, \pi/2)$, $r > 0$ the inclusion $M \setminus B_r(0) \subset \mathbb{C} \setminus S_\delta(a)$ is valid. Then for all $f \in L_2(\mathbb{R})$ and $\lambda \in M$ the estimate*

$$\|(\mathcal{H}_0 - \lambda)^{-1} f\|_{W_2^2(\mathbb{R})} \leqslant C \|f\|_{L_2(\mathbb{R})},$$

is valid where the constant C is independent of $\lambda \in M$.

Proof. Let $f \in L_2(\mathbb{R})$, $\lambda \in M$. Since $\lambda \notin \sigma(\mathcal{H}_0)$, it follows that the operator $(\mathcal{H}_0 - \lambda)^{-1} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is bounded and in accordance with formula (3.16) in [7, Ch. V, Sec. 3.5] the estimate

$$\|u\|_{L_2(\mathbb{R})} \leqslant \frac{\|f\|_{L_2(\mathbb{R})}}{\operatorname{dist}(\lambda, \sigma(\mathcal{H}_0))},$$

holds true for all $\lambda \in M$, where $u := (\mathcal{H}_0 - \lambda)^{-1} f$. The inequality (2.1) and the boundedness of the function q imply

$$\begin{aligned}
&(pu', u')_{L_2(\mathbb{R})} + (qu, u)_{L_2(\mathbb{R})} - \lambda \|u\|^2_{L_2(\mathbb{R})} = (f, u)_{L_2(\mathbb{R})}, \\
&\|u'\|^2_{L_2(\mathbb{R})} \leqslant C(1 + |\lambda|) \|u\|^2_{L_2(\mathbb{R})} + \|f\|_{L_2(\mathbb{R})} \|u\|_{L_2(\mathbb{R})},
\end{aligned}$$

where the constant C is independent of M. Now we express the second derivative of u by the equation $(\mathcal{H}_0 - \lambda)u = f$ and in view of two last inequalities obtain the estimates

$$\|(\mathcal{H}_0 - \lambda)^{-1} f\|_{W_2^2(\mathbb{R})} \leqslant C \frac{1 + |\lambda|}{\operatorname{dist}(\lambda, \sigma(\mathcal{H}_0))} \|f\|_{L_2(\mathbb{R})} \tag{3.1}$$

with the constant C independent of M. If the set M is bounded, the statement of the lemma follows from the obtained estimate. In the case the set M is unbounded the statement of the lemma follows from the estimate (3.1) and an obvious inequality

$$\sup_{M \setminus B_r(0)} \frac{1 + |\lambda|}{\operatorname{dist}(\lambda, \sigma(\mathcal{H}_0))} \leqslant \sup_{\mathbb{C} \setminus S_\delta(a)} \frac{1 + |\lambda|}{\operatorname{dist}(\lambda, \sigma(\mathcal{H}_0))} < \infty.$$

□

Lemma 3.2. *The operator $\mathcal{H}_\varepsilon$ is closed for all ε small enough.*

Proof. Since $(\mu_0^+ - 1) \notin \sigma(\mathcal{H}_0)$ by (2.2), Lemma 3.1 with $M = \{\mu_0^+ - 1\}$ and the uniform in ε boundedness of the operator $\mathcal{L}_\varepsilon$ yield

$$\begin{aligned}\|\mathcal{L}_\varepsilon v\|_{L_2(\mathbb{R})} = \|\mathcal{L}_\varepsilon v\|_{L_2(Q)} \leqslant & C\|v\|_{W_2^2(\mathbb{R})} \leqslant C\|(\mathcal{H}_0 - \mu_0^+ + 1)v\|_{L_2(\mathbb{R})} \\ \leqslant & C\left(\|\mathcal{H}_0 v\|_{L_2(\mathbb{R})} + |\mu_0^+ - 1|\|v\|_{L_2(\mathbb{R})}\right).\end{aligned}$$

Hence, the operator $\varepsilon\mathcal{L}_\varepsilon$ is $\mathcal{H}_0$-bounded and for ε small enough its $\mathcal{H}_0$-bound is strictly less than one. By [7, Ch. IV, Sec. 1.1, Thm. 1.1] it completes the proof. ☐

Proof of Theorem 2.1. We choose a pair of numbers $a > 0$, $\delta \in (0, \pi/2)$. It is sufficient to show that for ε small enough the inclusion $\sigma(\mathcal{H}_\varepsilon) \subset S_\delta(\mu_0^+ - a)$ is valid. In turn, this inclusion is equivalent to the existence of the resolvent of the operator $\mathcal{H}_\varepsilon$ for all $\lambda \in \mathbb{C} \setminus S_\delta(\mu_0^+ - a)$ if ε is small enough. Let us prove the last fact.

The set $M := \mathbb{C} \setminus S_\delta(\mu_0^+ - a)$ satisfies the hypothesis of Lemma 3.1, and this is why by this lemma and the uniform boundedness of $\mathcal{L}_\varepsilon$ we conclude that the operator $\mathcal{L}_\varepsilon(\mathcal{H}_0 - \lambda)^{-1}$ is bounded uniformly in ε. Thus, for ε small enough the operator $(\mathrm{I} - \varepsilon\mathcal{L}_\varepsilon(\mathcal{H}_0 - \lambda)^{-1})$ is boundedly invertible for all $\lambda \in M$. Employing this fact it is easy to check that the resolvent of $\mathcal{H}_\varepsilon$ exists for all $\lambda \in M$ and is given by the identity $(\mathcal{H}_\varepsilon - \lambda)^{-1} = (\mathcal{H}_0 - \lambda)^{-1}\big(\mathrm{I} - \varepsilon\mathcal{L}_\varepsilon(\mathcal{H}_0 - \lambda)^{-1}\big)^{-1}$. ☐

Let points x_0, x_1 be so that $\overline{Q} \subset (x_0, x_1)$. Without loss of generality we suppose that $(x_1 - x_0)$ is a natural number. By $\mathcal{H}_0^{(-)}$, $\mathcal{H}_0^{(0)}$, $\mathcal{H}_0^{(+)}$ we denote the operator

$$-\frac{d}{dx}p\frac{d}{dx} + q$$

respectively, in $L_2(-\infty, x_0)$, $L_2(x_0, x_1)$, $L_2(x_1, +\infty)$. As the domains for this operators we choose the subset of the functions from $W_2^2(-\infty, x_0)$, $W_2^2(x_0, x_1)$, $W_2^2(x_1, +\infty)$ vanishing, respectively, at x_0, at x_0, x_1, at x_1. Clearly, the operators $\mathcal{H}_0^{(-)}$, $\mathcal{H}_0^{(0)}$, $\mathcal{H}_0^{(+)}$ are self-adjoint.

Lemma 3.3. *The operator* $\big(\mathcal{H}_0^{(0)} - \mathrm{i}\big)^{-1} : L_2(x_0, x_1) \to L_2(x_0, x_1)$ *is compact.*

Proof. Since the operator $\mathcal{H}_0^{(0)}$ is self-adjoint, it follows that $\sigma(\mathcal{H}_0^{(0)}) \subset \mathbb{R}$. Thus the operator $\big(\mathcal{H}_0^{(0)} - \mathrm{i}\big)^{-1}$ is bounded as an operator in $L_2(x_0, x_1)$. By Banach theorem on the inverse operator [8, Ch. IV, Sec. 5.4, Thm. 3] we conclude that the operator $(\mathcal{H}_0^{(0)} - \mathrm{i})^{-1} : L_2(x_0, x_1) \to W_2^2(x_0, x_1)$ is bounded. Together with the compactness of the embedding $W_2^2(x_0, x_1)$ in $L_2(x_0, x_1)$ it completes the proof. ☐

Lemma 3.4. *The identity* $\sigma_{\mathrm{c}}\big(\mathcal{H}_0^{(-)} \oplus \mathcal{H}_0^{(+)}\big) = \sigma_{\mathrm{c}}(\mathcal{H}_0)$ *is valid.*

Proof. Let $W_2^2(\mathbb{R}, x_0)$ be the subset of the functions from $W_2^2(\mathbb{R})$ vanishing at x_0, and $\widetilde{\mathcal{H}}_0$ be the restriction of $\mathcal{H}_0$ on $W_2^2(\mathbb{R}, x_0)$. Then the operator $\mathcal{H}_0$ is a closed extension of $\widetilde{\mathcal{H}}_0$. It is obvious that the factor-space $W_2^2(\mathbb{R})/W_2^2(\mathbb{R}, x_0)$ is

one-dimensional and $\mathcal{H}_0$ is hence a finite-dimensional extension of $\widetilde{\mathcal{H}}_0$. By [1, Ch. I, Sec. 1.2, Thm. 4] it follows that

$$\sigma_c(\widetilde{\mathcal{H}}_0) = \sigma_c(\mathcal{H}_0). \tag{3.2}$$

The operators $\widetilde{\mathcal{H}}_0$ and $\mathcal{H}_0^{(-)} \oplus \mathcal{H}_0^{(+)}$ are unitarily equivalent; the corresponding unitary operator is defined as

$$\big(\mathcal{U}\big(u^{(-)}, u^{(+)}\big)\big)(x) := \begin{cases} u^{(-)}(x), & x < x_0, \\ u^{(+)}(x), & x > x_1, \end{cases}$$

where $u^{(\pm)}$ belong to the domains of $\mathcal{H}_0^{(\pm)}$. Therefore, $\sigma_c\left(\mathcal{H}_0^{(-)} \oplus \mathcal{H}_0^{(+)}\right) = \sigma_c(\widetilde{\mathcal{H}}_0)$, that together with (3.2) completes the proof. $\square$

Proof of Theorem 2.2. Let $\mathcal{H}_\varepsilon^{(0)} := \mathcal{H}_0^{(0)} - \varepsilon\mathcal{L}_\varepsilon$ be an operator in $L_2(x_0, x_1)$ whose domain coincides with one of $\mathcal{H}_0^{(0)}$. Here the operator $\mathcal{L}_\varepsilon$ is defined in the space $L_2(x_0, x_1)$ by the same scheme as one used when defining this operator in the space $L_2(\mathbb{R})$. By analogy with the proof of Lemma 3.2 one can make sure that the operator $\mathcal{H}_\varepsilon^{(0)}$ is closed for all ε small enough. The operator $\mathcal{H}_\varepsilon$ is a finite-dimensional extension of $\mathcal{H}_0^{(-)} \oplus \mathcal{H}_\varepsilon^{(0)} \oplus \mathcal{H}_0^{(+)}$. Hence, by [1, Ch. I, Sec. 1.2, Thm. 4] the identity $\sigma_c(\mathcal{H}_\varepsilon) = \sigma_c\left(\mathcal{H}_0^{(-)} \oplus \mathcal{H}_0^{(+)}\right) \cup \sigma_c(\mathcal{H}_\varepsilon^{(0)})$ is valid. It is also follows from [1, Ch. I, Sec. 1.1, 1.2] that $\sigma_r(\mathcal{H}_\varepsilon) \subseteq \sigma_r(\mathcal{H}_0^{(+)} \oplus \mathcal{H}_\varepsilon^{(0)} \oplus \mathcal{H}_0^{(+)}) = \sigma_r(\mathcal{H}_0^{(+)}) \cup \sigma_r(\mathcal{H}_\varepsilon^{(0)}) \cup \sigma_r(\mathcal{H}_0^{(+)})$. The operators $\mathcal{H}_0^{(\pm)}$ are self-adjoint, and thus $\sigma_r(\mathcal{H}_0^{(-)}) = \sigma_r(\mathcal{H}_0^{(+)}) = \emptyset$. In view of Lemma 3.4 it sufficient to show that $\sigma_c(\mathcal{H}_\varepsilon^{(0)}) = \sigma_r(\mathcal{H}_\varepsilon^{(0)}) = \emptyset$. Let us prove these identities.

By Lemma 3.3 we obtain in turn that for all ε small enough the operator $\big(\mathrm{I} - \varepsilon\mathcal{L}_\varepsilon(\mathcal{H}_0^{(0)} - \mathrm{i})^{-1}\big)^{-1} : L_2(x_0, x_1) \to L_2(x_0, x_1)$ is bounded and due to

$$(\mathcal{H}_\varepsilon^{(0)} - \mathrm{i})^{-1} = (\mathcal{H}_0^{(0)} - \mathrm{i})^{-1}\big(\mathrm{I} - \varepsilon\mathcal{L}_\varepsilon(\mathcal{H}_0^{(0)} - \mathrm{i})^{-1}\big)^{-1}$$

the resolvent $(\mathcal{H}_\varepsilon^{(0)} - \mathrm{i})^{-1}$ is a compact operator in $L_2(x_0, x_1)$. By [7, Ch. III, Sec. 6.8, Thm. 6.29] it follows that the spectrum of $\mathcal{H}_\varepsilon^{(0)}$ consists of at most countably many eigenvalues of finite multiplicity and thus $\sigma_c(\mathcal{H}_\varepsilon^{(0)}) = \sigma_r(\mathcal{H}_\varepsilon^{(0)}) = \emptyset$. $\square$

4 Proof of Theorem 2.4

By $\rho(\lambda)$ and $\varkappa(\lambda)$ we denote the multiplier and the quasi-momentum multiplied by $-\mathrm{i}$ corresponding to the equation (2.6):

$$\rho(\lambda) := \frac{D(\lambda) + \sqrt{D^2(\lambda) - 4}}{2}, \quad \varkappa(\lambda) := \ln \rho(\lambda). \tag{4.1}$$

The branch of the root is specified by the requirement $|\rho(\lambda)| \geqslant 1$. In the case $|\rho(\lambda)| = 1$, the concrete choice of the branch is not important. The branch of logarithm is specified by $\ln 1 = 0$.

In accordance with Floquet-Lyapunov theorem the equation (2.6) has a fundamental system,

$$\varphi_1(x,\lambda) = \mathrm{e}^{\varkappa(\lambda)x}\Phi_1(x,\lambda), \quad \varphi_2(x,\lambda) = \mathrm{e}^{-\varkappa(\lambda)x}\Phi_2(x,\lambda),$$

where $\Phi_i(\cdot,\lambda)$ are 1-periodic functions. These formulas are valid if $\rho(\lambda) \neq \pm 1$, as well as in the case $\rho(\lambda) = \pm 1$, $|\theta_1(\lambda)|^2 + |\theta_2'(\lambda)|^2 \neq 0$. If $\rho(\lambda) = \pm 1$, $\theta_1(\lambda) = \theta_2'(\lambda) = 0$, a fundamental system of the equation (2.6) reads as follows

$$\varphi_1(x,\lambda) = \mathrm{e}^{\varkappa(\lambda)x}\Phi_1(x,\lambda), \quad \varphi_2(x,\lambda) = \mathrm{e}^{\varkappa(\lambda)x}\big(x\Phi_1(x,\lambda) + \Phi_2(x,\lambda)\big),$$

where $\Phi_i(\cdot,\lambda)$ are 1-periodic functions.

We indicate by $W[f_1,f_2]$ the Wronskian of the functions $f_1 = f_1(x)$, $f_2 = f_2(x)$.

Lemma 4.1. *For any $\alpha_1,\alpha_2 \in \mathbb{R}$ the functions $\theta_i(x,\lambda)$, $i = 1,2$, are holomorphic w.r.t. λ in the norm of $C^2[\alpha_1,\alpha_2]$. The function $D(\lambda)$ is holomorphic. The branches of the function $\rho(\lambda)$ are holomorphic everywhere in the complex plane except the edges of non-degenerate lacunas in the spectrum of $\mathcal{H}_0$, those are branching points for this function. The identity*

$$W[\theta_1,\theta_2](x) = \frac{1}{p(x)} \tag{4.2}$$

is valid.

Proof. The identity (4.2) follows from the initial conditions (2.7) and the Liouville formula for the Wronskian.

In view of $\mathcal{P}(\lambda,\alpha)$ we indicate an integral operator

$$\big(\mathcal{P}(\lambda,\alpha)f\big)(x) := \int\limits_{\alpha}^{x} \big(\theta_1(x,\lambda)\theta_2(t,\lambda) - \theta_1(t,\lambda)\theta_2(x,\lambda)\big)f(t)\,\mathrm{d}t. \tag{4.3}$$

By (4.2) this operator determines the solution to a Cauchy problem

$$\left(-\frac{d}{dx}p\frac{d}{dx} + q\right)v = f, \quad x \in \mathbb{R}, \qquad v(\alpha) = \frac{dv}{dx}(\alpha) = 0.$$

It is easy to make sure that for all $\lambda \in \mathbb{C}$ a linear operator $\mathcal{P} : C[\alpha_1,\alpha_2] \to C^2[\alpha_1,\alpha_2]$ is bounded for any $\alpha_1 \leqslant \alpha \leqslant \alpha_2$. One can check that the functions $\theta_i(x,\lambda)$ are solutions to the equation

$$\big(\mathrm{I} - \lambda\mathcal{P}(0,0)\big)\theta_i(\cdot,\lambda) = \theta_i(\cdot,0), \tag{4.4}$$

which we regard as one in $C[\alpha_1,\alpha_2]$. This is the Volterra equation. Hence the operator $\mathrm{I} - \lambda\mathcal{P}(0,0)$ is boundedly invertible for all $\lambda \in \mathbb{C}$ (see, for instance, [8, Ch. XI, Sec. 3.3]). By [9, Ch. XI, Sec. 4, Proposition 4.5] it follows that the operator $\big(\mathrm{I} - \lambda\mathcal{P}(0,0)\big)^{-1}$ is boundedly holomorphic w.r.t. $\lambda \in \mathbb{C}$ as an operator

in $C[\alpha_1,\alpha_2]$. Therefore, the functions $\theta_i(x,\lambda)$ are holomorphic w.r.t. $\lambda\in\mathbb{C}$ in the norm of $C[\alpha_1,\alpha_2]$. Since by (4.4) the relations

$$\theta_i(x,\lambda)=\theta_i(x,0)+\lambda\big(\mathcal{P}(0,0)\theta_i(\cdot,\lambda)\big)(x,\lambda),$$

hold true, we infer that the functions $\theta_i(x,\lambda)$ are holomorphic w.r.t. $\lambda\in\mathbb{C}$ in the norm of $C^2[\alpha_1,\alpha_2]$ as well. The holomorphy of the function $D(\lambda)$ is implied by one of $\theta_i(x,\lambda)$. The identities $D(\lambda)^2=4$ hold true only for $\lambda=\mu_n^\pm$ (see [2, Ch. 2, Sec. 2.3, Thm. 2.3.1]), this is why only these points can be branching points of the function ρ. Item c) of the proof of Theorem 2.3.1 in [2, Ch. 2, Sec. 2.3] implies that if $\mu_n^\pm$ is an edge of a non-degenerate lacuna, it follows that $(D^2-4)'|_{\lambda=\mu_n^\pm}\neq 0$. If a lacuna degenerates ($\mu_n^-=\mu_n^+$), the relations $(D^2-4)'|_{\lambda=\mu_n^\pm}=0$, $(D^2-4)''|_{\lambda=\mu_n^\pm}\neq 0$ hold true. This fact implies the statement of the lemma on ρ. □

Proof of Theorem 2.4. Suppose that $\lambda\in\sigma_{\mathrm{c}}(\mathcal{H}_0)\cap K$ is an eigenvalue of the operator $\mathcal{H}_\varepsilon$ for some ε small enough. An associated eigenfunction satisfies the equation

$$\left(-\frac{d}{dx}p\frac{d}{dx}+q-\lambda-\varepsilon\mathcal{L}_\varepsilon\right)\psi=0. \tag{4.5}$$

For $x\notin Q$ this equation coincides with the equation in (2.6). Hence, for x lying to the left w.r.t. the set Q, the function ψ reads as follows,

$$\psi(x)=c_1\varphi_1(x,\lambda)+c_2\varphi_2(x,\lambda),$$

where c_i are constants. The similar behaviour is valid for x lying to the right w.r.t. Q. Since $\lambda\in\sigma_{\mathrm{c}}(\mathcal{H}_0)$, due to Item (v) of Theorem 2.3.1 in [2, Ch. 2, Sec. 2.3] the identity $|\rho(\lambda)|=1$ is valid, and hence $\operatorname{Re}\varkappa(\lambda)=0$. It follows that the functions $\varphi_i(x,\lambda)$ are not square integrable at infinity. The function ψ is thus an element of $L_2(\mathbb{R})$ only if

$$\psi\equiv 0,\quad x\notin Q. \tag{4.6}$$

Since $\psi\in W_2^2(\mathbb{R})\subset C^1(\mathbb{R})$, we have

$$\psi(x_0)=\psi'(x_0)=0. \tag{4.7}$$

We remind that the points x_0, x_1 are so that $\overline{Q}\subset(x_0,x_1)$. By the definition (4.3) of $\mathcal{P}$ the initial problem (4.5), (4.7) is equivalent to an integral equation

$$\big(\mathrm{I}-\varepsilon\mathcal{L}_\varepsilon\mathcal{P}(\lambda,x_0)\big)\psi=0. \tag{4.8}$$

The integral operator $\mathcal{P}$ is a linear bounded operator from $L_2(Q)$ into $W_2^2(Q)$. Moreover, by Lemma 4.1 it is bounded uniformly in $\lambda\in K$. This fact and the uniform in ε boundedness of $\mathcal{L}_\varepsilon$ yield that the operator $\mathcal{L}_\varepsilon\mathcal{P}(\lambda,x_0):L_2(Q)\to L_2(Q)$ is bounded uniformly in ε and $\lambda\in K$. Thus, for ε small enough and $\lambda\in K$ the operator $\big(\mathrm{I}-\varepsilon\mathcal{L}_\varepsilon\mathcal{P}(\lambda,x_0)\big)$ is boundedly invertible, and the equation (4.8) therefore has the trivial solution only. Hence, $\psi\equiv 0$ for $x\in Q$. In view of (4.6) it follows that the function ψ is identically zero. It contradicts to the assumption that ψ is an eigenfunction. □

5 Proof of Theorems 2.3, 2.6

Consider the equation

$$\left(-\frac{d}{dx}p\frac{d}{dx}+q-\varepsilon\mathcal{L}_\varepsilon-\lambda\right)u=f,\quad x\in\mathbb{R},\tag{5.1}$$

where $f\in L_2(\mathbb{R};(x_0,x_1))$, $L_2(\mathbb{R};(x_0,x_1))$ is a subset of the functions in $L_2(\mathbb{R})$ having supports inside $[x_0,x_1]$. We are looking for the solutions to this equation satisfying the conditions

$$\begin{aligned}&u(x,\lambda)=\mathrm{e}^{-\varkappa(\lambda)x}\Phi_+(x,\lambda),\quad x\geqslant x_1,\\&u(x,\lambda)=\mathrm{e}^{\varkappa(\lambda)x}\Phi_-(x,\lambda),\quad x\leqslant x_0,\end{aligned}\tag{5.2}$$

where $\Phi_\pm$ are 1-periodic in x functions, and the branch of the logarithm in the definition of $\varkappa(\lambda)$ is specified by the relation $\ln 1=0$. Here the branch of the function ρ is not specified yet. We will study the dependence of the solution to (5.1), (5.2) of λ, which allows us to prove Theorems 2.3, 2.6. In order to solve the problem (5.1), (5.2) we employ the scheme suggested in [9, Ch. XIV, Sec. 4].

We set

$$\begin{aligned}G(x,t,\lambda):=&\frac{1}{\rho(\lambda)-\rho^{-1}(\lambda)}\big(\theta_2(\lambda)\theta_1(t,\lambda)\theta_1(x,\lambda)-(\rho(\lambda)-\theta_2'(\lambda))\theta_2(t,\lambda)\theta_1(x,\lambda)\\&+(\rho(\lambda)-\theta_1(\lambda))\theta_1(t,\lambda)\theta_2(x,\lambda)-\theta_1'(\lambda)\theta_2(t,\lambda)\theta_2(x,\lambda)\big),\quad t\geqslant x\\G(x,t,\lambda):=&\frac{1}{\rho(\lambda)-\rho^{-1}(\lambda)}\big(\theta_2(\lambda)\theta_1(t,\lambda)\theta_2(x,\lambda)-(\rho(\lambda)-\theta_2'(\lambda))\theta_1(t,\lambda)\theta_2(x,\lambda)\\&+(\rho(\lambda)-\theta_1(\lambda))\theta_2(t,\lambda)\theta_1(x,\lambda)-\theta_1'(\lambda)\theta_2(t,\lambda)\theta_2(x,\lambda)\big),\quad x\geqslant t.\end{aligned}$$

The function $G(x,t,\lambda)$ is well-defined for all $\lambda\in\mathbb{C}$ except the edges of the non-degenerate lacunas in the spectrum of $\mathcal{H}_0$. Indeed, if $\lambda\neq\mu_n^\pm$, it follows that $\rho^2(\lambda)\neq 1$ ([2, Ch. 2, Sec. 2.3, Thm. 2.3.1]), and thus $\rho(\lambda)\neq\rho^{-1}(\lambda)$. If lacuna degenerates ($\mu_n^-=\mu_n^+$), by Item c) of the proof of Theorem 2.3.1 in [2, Ch. 2, Sec. 2.3] the identities

$$\begin{aligned}&D(\lambda)=(-1)^n\big(2-\gamma(\lambda-\mu)^2\big)+\mathcal{O}\big(|\lambda-\mu|^3\big),\\&\rho(\lambda)=(-1)^n\big(1\pm\mathrm{i}\sqrt{\gamma}(\lambda-\mu)\big)+\mathcal{O}\big(|\lambda-\mu|^2\big),\\&\theta_1(\lambda)=(-1)^n+\dot\theta_1(\mu)(\lambda-\mu)+\mathcal{O}\big(|\lambda-\mu|^2\big),\\&\theta_2(\lambda)=\dot\theta_2(\mu)(\lambda-\mu)+\mathcal{O}\big(|\lambda-\mu|^2\big),\\&\theta_1'(\lambda)=\dot\theta_1'(\mu)(\lambda-\mu)+\mathcal{O}\big(|\lambda-\mu|^2\big),\end{aligned}\tag{5.3}$$

are valid, where $\lambda\to\mu:=\mu_n^-=\mu_n^+$, and $\gamma>0$ is a constant. The sign "$\pm$" in the identity for $\rho(\lambda)$ corresponds to the different branches of this function. These identities imply that there exists a finite limit of the function G as $\lambda\to\mu$, which we regard as a definition of this function at $\lambda=\mu$.

On the functions $f \in L_2(\mathbb{R}; (x_0, x_1))$ we introduce the operator $\mathcal{G}(\lambda)$ with the kernel $G(x, t, \lambda)$:

$$\big(\mathcal{G}(\lambda)f\big)(x, \lambda) := \int\limits_{\mathbb{R}} G(x, t, \lambda) f(t)\, \mathrm{d}t.$$

It is clear that it is bounded as an operator from $L_2(x_0, x_1)$ into $W_2^2(x_0, x_1)$ for all values of λ not coinciding with the edges of non-degenerate lacunas in the spectrum of $\mathcal{H}_0$.

Bearing in mind the definition of $\rho(\lambda)$ and $D(\lambda)$ by direct calculations we check that the function G is the Green function for the equation

$$\left(-\frac{d}{dx} p \frac{d}{dx} + q - \lambda\right) v = f, \quad x \in \mathbb{R}, \tag{5.4}$$

and $v := \mathcal{G}(\lambda) f$, where $f \in L_2(\mathbb{R}; (x_0, x_1))$, is a solution to this equation.

Taking into account the initial conditions (2.7) and the periodicity of p and q, one can check easily that

$$\theta_i(x + 1, \lambda) = \theta_i(1, \lambda)\theta_1(x, \lambda) + \theta_i'(1, \lambda)\theta_2(x, \lambda), \quad i = 1, 2. \tag{5.5}$$

Employing this relation, the identities (4.2) and the formula

$$\rho(\lambda) + \frac{1}{\rho(\lambda)} = \theta_1(\lambda) + \theta_2'(\lambda), \tag{5.6}$$

following from (4.1), it is easy to make sure that the function v obeys the identities

$$v(x + 1, \lambda) = \frac{1}{\rho(\lambda)} v(x, \lambda), \quad x \geqslant x_1, \quad v(x - 1, \lambda) = \frac{1}{\rho(\lambda)} v(x, \lambda), \quad x \leqslant x_0. \tag{5.7}$$

It implies that the function v satisfies the conditions (5.2).

Let $\zeta = \zeta(x)$ be an infinitely differentiable cut-off function vanishing for $x \notin [x_0, x_1]$ and equalling one in a neighbourhood of the segment $\overline{Q}$, and $g \in L_2(\mathbb{R}; (x_0, x_1))$ be a function. We denote

$$v := \mathcal{G}(\lambda) g, \quad w_\varepsilon := (\mathcal{H}_\varepsilon^{(0)} - \mathrm{i})^{-1} \mathcal{L}_\varepsilon v. \tag{5.8}$$

where, we remind, $\mathcal{H}_\varepsilon^{(0)}$ is an operator introduced in the proof of Theorem 2.2. This operator is bounded as one from the subspace of the functions in $W_2^2(x_0, x_1)$ vanishing at x_0, x_1, into $L_2(x_0, x_1)$. As it was established in the proof of Theorem 2.2, the operator $(\mathcal{H}_\varepsilon^{(0)} - \mathrm{i})^{-1}$ exists for all ε small enough, and Banach theorem on inverse operator [8, Ch. IV, Sec. 5.4, Thm. 3] thus implies that for ε small enough the operator $(\mathcal{H}_\varepsilon^{(0)} - \mathrm{i})^{-1} : L_2(x_0, x_1) \to W_2^2(x_0, x_1)$ is bounded. It is clear that it is bounded uniformly in ε. Thus, an operator $(\mathcal{H}_\varepsilon^{(0)} - \mathrm{i})^{-1} \mathcal{L}_\varepsilon$ is bounded uniformly in ε as one in $W_2^2(x_0, x_1)$.

We construct the solution to the problem (5.1), (5.2) as follows

$$u(x,\lambda) := v(x,\lambda) + \varepsilon\zeta(x)w_\varepsilon(x,\lambda). \tag{5.9}$$

This functions satisfies the conditions (5.2). We substitute it into the left hand side of (5.1) to obtain

$$\begin{aligned}\left(-\frac{d}{dx}p\frac{d}{dx} + q - \varepsilon\mathcal{L}_\varepsilon - \lambda\right)(v + \varepsilon\zeta w_\varepsilon) &= g - \varepsilon\mathcal{L}_\varepsilon v + \varepsilon\mathcal{T}_\varepsilon(\lambda)g \\ &+ \varepsilon\zeta\left(-\frac{d}{dx}p\frac{d}{dx} + q - \varepsilon\mathcal{L}_\varepsilon - \mathrm{i}\right)w_\varepsilon = g + \mathcal{T}_\varepsilon(\lambda)g, \\ \mathcal{T}_\varepsilon(\lambda)g := -\frac{d}{dx}pw_\varepsilon\frac{d\zeta}{dx} - p\frac{d\zeta}{dx}\frac{dw_\varepsilon}{dx} &+ (\mathrm{i}-\lambda)\zeta w_\varepsilon.\end{aligned}$$

Here we have also employed the relation $\mathcal{L}_\varepsilon\zeta w_\varepsilon = \mathcal{L}_\varepsilon w_\varepsilon = \zeta\mathcal{L}_\varepsilon w_\varepsilon$ which follows from the identity $\zeta \equiv 1$, $x \in \overline{Q}$. Thus, the function u defined by (5.9) is a solution to (5.1) if

$$g + \varepsilon\mathcal{T}_\varepsilon(\lambda)g = f. \tag{5.10}$$

Lemma 5.1. *The problem (5.1), (5.2) is equivalent to the equation (5.10) for all $\lambda \in \mathbb{C}$ not coinciding with the edges of the non-degenerate lacunas in the spectrum of $\mathcal{H}_0$. Namely, for each solution of (5.10) there exists the unique solution to the problem (5.1), (5.2) defined by (5.8), (5.9). For each solution u of the problem (5.1), (5.2) there exists the unique function g satisfying the equation (5.10) and related to u by (5.8), (5.9).*

Proof. If g is a solution to (5.10), as it was shown above, it follows that the function u introduced by (5.8), (5.9) is a solution to (5.1), (5.2).

Let u be a solution to (5.1), (5.2). We define the functions v, w_ε and g as

$$w_\varepsilon := (\mathcal{H}_0^{(0)} - \mathrm{i})^{-1}\mathcal{L}_\varepsilon u, \quad v := u - \varepsilon\zeta w_\varepsilon, \quad g := \left(-\frac{d}{dx}p\frac{d}{dx} + q - \lambda\right)v.$$

The function v satisfies the relations (5.2) and hence the former of the formulas (5.8) holds true. The identity (5.9) is obviously to be valid. Since

$$(\mathcal{H}_0^{(0)} - \varepsilon\mathcal{L}_\varepsilon - \mathrm{i})w_\varepsilon = \mathcal{L}_\varepsilon u - \varepsilon\mathcal{L}_\varepsilon w_\varepsilon = \mathcal{L}_\varepsilon(u - \varepsilon\zeta w_\varepsilon) = \mathcal{L}_\varepsilon v,$$

the latter of the formulas (5.8) holds true as well. The definition of the functions g and v and the equation (5.1) imply that

$$\begin{aligned}g = \left(-\frac{d}{dx}p\frac{d}{dx} + q - \lambda\right)(u - \varepsilon\zeta w) &= f + \varepsilon\mathcal{L}_\varepsilon u - \varepsilon\mathcal{T}_\varepsilon(\lambda)g - \varepsilon\zeta(\mathcal{H}_0^{(0)} - \mathrm{i})w_\varepsilon \\ = f + \varepsilon\mathcal{L}_\varepsilon u - \varepsilon\mathcal{T}_\varepsilon(\lambda)g - \varepsilon\zeta\mathcal{L}_\varepsilon u &= f - \varepsilon\mathcal{T}_\varepsilon(\lambda)g,\end{aligned}$$

which yields the equation (5.10). $\square$

The properties of $\mathcal{G}$ and $(\mathcal{H}_\varepsilon^{(0)} - \mathrm{i})^{-1}\mathcal{L}_\varepsilon$ established above allow us to claim that the mapping $g \mapsto w_\varepsilon$ is a linear operator from $L_2(x_0, x_1)$ into $W_2^1(x_0, x_1)$ bounded uniformly in ε. Therefore, the operator $\mathcal{T}_\varepsilon(\lambda) : L_2(x_0, x_1) \to W_2^1(x_0, x_1)$ is bounded uniformly in ε. Moreover, it follows that the operator $\mathcal{T}_\varepsilon$ is compact as one in $L_2(x_0, x_1)$.

Proof of Theorem 2.6. For any $\delta > 0$ we denote $K_\delta := K \setminus \bigcup\limits_n \big(B_\delta(\mu_n^-) \cup B_\delta(\mu_n^+)\big)$, where the union is taken over the edges of the non-degenerate lacunas in the spectrum of $\mathcal{H}_0$. For $\lambda \in K_\delta$ we specify the branch of the root in the definition (4.1) of the function ρ by the condition $|\rho(\lambda)| \geqslant 1$; in the case $|\rho(\lambda)| = 1$ the exact choice of the branch is inessential. The operator $\mathcal{G}(\lambda)$ is piecewise continuous w.r.t. $\lambda \in K_\delta$. Therefore, the operator $\mathcal{T}_\varepsilon(\lambda)$ is bounded uniformly in ε and $\lambda \in K_\delta$ as an operator in $L_2(x_0, x_1)$. By this fact we infer that for ε small enough the operator $\mathrm{I} + \varepsilon\mathcal{T}_\varepsilon(\lambda)$ is boundedly invertible for all $\lambda \in K_\delta$, and thus the equation

$$g + \varepsilon\mathcal{T}_\varepsilon(\lambda)g = 0 \tag{5.11}$$

has no nontrivial solutions for $\lambda \in K_\delta$ and ε small enough. By Lemma 5.1 it implies that the problem (4.5), (5.2) has no nontrivial solutions. Since $\operatorname{Re} \varkappa(\lambda) \geqslant 0$, $\lambda \in K_\delta$, by the choice of the branch of ρ, the equation (4.5) has no nontrivial solutions in the space $W_2^2(\mathbb{R})$. Therefore, the operator $\mathcal{H}_\varepsilon$ has no eigenvalues in K_δ if ε is small enough. This fact and the arbitrary choice of δ complete the proof. $\square$

Lemma 5.2. *The point spectrum of $\mathcal{H}_\varepsilon$ consist of at most countably many eigenvalues. The edges of non-degenerate lacunas in the spectrum of $\mathcal{H}_0$ are the only possible finite accumulation points for these eigenvalues. Each eigenvalue not coinciding with such edge is of finite multiplicity.*

Proof. For any $\delta > 0$ we indicate $M := \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \geqslant 0\} \setminus \bigcup\limits_n \big(B_\delta(\mu_n^-) \cup B_\delta(\mu_n^+)\big)$. Here the union is taken over n corresponding to the edges of the non-degenerate lacunas in the spectrum of $\mathcal{H}_0$. The statement of Lemma 4.1 on ρ implies that the set M can be covered as follows, $M \subset \widetilde{M} := \bigcup\limits_{j=1}^{\infty} M_j$, where M_j are simply connected domains such that $M_j \setminus S_{\pi/4}(\mu_0^+ - 1) \neq \emptyset$, and for each of them one can choose the branch of the function ρ to be holomorphic w.r.t. $\lambda \in M_j$, and to obey the estimate $|\rho(\lambda)| \geqslant 1$ for $\operatorname{Im} \lambda \geqslant 0$. Lemma 4.1 and (5.3) yield that the operator $\mathcal{G}(\lambda) : L_2(x_0, x_1) \to W_2^2(x_0, x_1)$ is holomorphic w.r.t. $\lambda \in M_j$. This fact and the properties of the operator $(\mathcal{H}_\varepsilon^{(0)} - \mathrm{i})^{-1}\mathcal{L}_\varepsilon$ established above follow that the operator $\mathcal{T}_\varepsilon(\lambda)$ is holomorphic in $\lambda \in M_j$ as an operator in $L_2(x_0, x_1)$.

Let $\lambda \in M$ be an eigenvalue of the operator $\mathcal{H}_\varepsilon$, then $\lambda \in M_j$ for some j. The associated eigenfunction is a solution to the problem (4.5), (5.2), where $\varkappa$ is defined via the branch of ρ corresponding to M_j. Due to Lemma 5.1 it means that the corresponding equation (5.11) has a nontrivial solution. The compactness of

the operator $\mathcal{T}_\varepsilon(\lambda)$ implies that the number of such linear independent solutions is finite and λ is thus an eigenvalue of finite multiplicity. Since $M_j \setminus S_{\pi/4}(\mu_0^+) \neq \emptyset$, in accordance with Theorem 2.1 there exists a point $\lambda_* \in M_j \setminus \sigma(\mathcal{H}_\varepsilon)$. Therefore, the equation (5.11) has no nontrivial solution for $\lambda = \lambda_*$ that together with the compactness of $\mathcal{T}_\varepsilon(\lambda_*)$ implies the bounded invertibility of $\mathrm{I}+\varepsilon\mathcal{T}_\varepsilon(\lambda_*)$. This fact and the above established holomorphy of $\mathcal{T}_\varepsilon$ in $\lambda \in M_j$ allow us to employ Theorem 7.1 in [9, Ch. XV, Sec. 7] and to conclude that the operator $(\mathrm{I}+\varepsilon\mathcal{T}_\varepsilon)^{-1}$ is meromorphic in $\lambda \in M_j$ and has at most countably many poles in M_j those can accumulate at the boundary points of M_j only. Moreover, the poles of $(\mathrm{I}+\varepsilon\mathcal{T}_\varepsilon)^{-1}$ are values of λ for which the equation (5.11) has a nontrivial solution. Therefore, the eigenvalues of $\mathcal{H}_\varepsilon$ lying in $M_j \cap \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geqslant 0\}$ are poles of $(\mathrm{I}+\varepsilon\mathcal{T}_\varepsilon)^{-1}$ corresponding to M_j. Thus, the operator $\mathcal{H}_\varepsilon$ has at most countably many eigenvalues in M. The points of M can not be accumulation points for these eigenvalues since each such point is inner for one of the sets M_j. In the same way one can prove that the set obtained from M by mirror symmetry w.r.t. real axis contains at most finitely many eigenvalues of $\mathcal{H}_\varepsilon$ of finite multiplicity those have no accumulation points inside this set. The number δ being arbitrary completes the proof. $\square$

In view of this lemma it remains to prove that edges of the non-degenerate lacunas in the spectrum of $\mathcal{H}_0$ are not the accumulation points for the eigenvalues of $\mathcal{H}_\varepsilon$. We should also show that in the case such an edge is an eigenvalue of $\mathcal{H}_\varepsilon$ it is of finite multiplicity. We will prove these facts on the basis of the equation similar to (5.10). We can not employ exactly this equation since the function ρ has branching points at $\mu_n^\pm$, and hence the operator $\mathcal{T}_\varepsilon$ is not holomorphic at these points.

First we prove an auxiliary statement.

Lemma 5.3. *Let $\mu_n^\pm$ be an edge of a non-degenerate lacuna in the spectrum of $\mathcal{H}_0$. Then*

1. *At least one of the numbers $\theta_1'(\mu_n^+)$ and $\theta_2(\mu_n^+)$ (respectively, $\theta_1'(\mu_n^-)$ and $\theta_2(\mu_n^-)$) is non zero and the inequality $\theta_1'(\mu_n^\pm)\theta_2(\mu_n^\pm) \leqslant 0$ holds true.*
2. *$D(\mu_n^\pm) = 2(-1)^n$, $\mp(-1)^n\dot{D}(\mu_n^\pm) > 0$.*
3. *The numbers $\dot{D}(\mu_n^\pm)$ are given by*

$$\dot{D}(\mu_n^\pm) = \int\limits_0^1 \Big(\theta_1'(\mu_n^\pm)\theta_2^2(x,\mu_n^\pm) + (\theta_1(\mu_n^\pm) - \theta_2'(\mu_n^\pm))\theta_1(x,\mu_n^\pm)\theta_2(x,\mu_n^\pm) - \\ - \theta_2(\mu_n^\pm)\theta_1^2(x,\mu_n^\pm)\Big)^2 \,\mathrm{d}x.$$

Moreover,

$$\dot{D}(\mu_n^\pm) = -\frac{1}{4\theta_2(\mu_n^\pm)} \int\limits_0^1 \Big(2\theta_2(\mu_n^\pm)\theta_1(x,\mu_n^\pm) + (\theta_1(\mu_n^\pm) - \theta_2'(\mu_n^\pm))\theta_2(x,\mu_n^\pm)\Big)^2 \,\mathrm{d}x,$$

if $\theta_2(\mu_n^\pm)\neq 0$, *and*

$$\dot{D}(\mu_n^\pm)=\frac{1}{4\theta_1'(\mu_n^\pm)}\int\limits_0^1\Big(2\theta_1'(\mu_n^\pm)\theta_2(x,\mu_n^\pm)+(\theta_1(\mu_n^\pm)-\theta_2'(\mu_n^\pm))\theta_1(x,\mu_n^\pm)\Big)^2\,\mathrm{d}x,$$

if $\theta_1'(\mu_n^\pm)\neq 0$.

The lemma follows from Theorem 2.3.1 in [2, Ch. 2, Sec. 2.3], and the formula (2.3.7) and the relation $D^2(\lambda)=4+(\theta_1(\lambda)-\theta_2'(\lambda))^2+4\theta_1'(\lambda)\theta_2(\lambda)$ established in the proof of this theorem.

Let $\mu_n^\pm$ be an edge of a non-degenerate lacuna in the spectrum of $\mathcal{H}_0$. In a small neighbourhood of $\mu_n^\pm$ we introduce a new complex parameter by the rule $\lambda:=\mu_n^\pm\mp k^2$. We denote

$$\rho_n^\pm=\rho_n^\pm(k):=\frac{D(\mu_n^\pm\mp k^2)+(-1)^n\sqrt{D^2(\mu_n^\pm\mp k^2)-4}}{2},\tag{5.12}$$

where the branch of the root is specified by the condition $\sqrt{1}=1$, if $\arg k\in[0,\pi)$ and $\sqrt{1}=-1$, if $\arg k\in[\pi,2\pi)$. By Item 2 of Lemma 5.3 the identities

$$D(\mu_n^\pm\mp k^2)=(-1)^n(2+|\dot{D}(\mu_n^\pm)|k^2)+\mathcal{O}(|k|^4),\quad D^2(\mu_n^\pm\mp k^2)-4=4|\dot{D}(\mu_n^\pm)|k^2+\mathcal{O}(|k|^4)$$

are valid for k small enough. Thus, the function $\rho_n^\pm$ is holomorphic in k and its Taylor expansion reads as follows,

$$\rho_n^\pm(k)=(-1)^n\left(1+\sqrt{|\dot{D}(\mu_n^\pm)|}k+\frac{1}{2}|\dot{D}(\mu_n^\pm)|k^2\right)+\mathcal{O}(|k|^3).\tag{5.13}$$

We set

$$\begin{aligned}\varkappa_n&=\varkappa_n^\pm(k):=\ln\rho_n^\pm(k), &&\text{if } n \text{ is even},\\ \varkappa_n&=\varkappa_n^\pm(k):=\ln\rho_n^\pm(k)-\pi\mathrm{i}, &&\text{if } n \text{ is odd},\end{aligned}$$

where the branch of the logarithm is specified by $\ln 1=0$. For k small enough the function $\varkappa_n^\pm(k)$ is holomorphic w.r.t. k and the identity

$$\varkappa_n^\pm(k)=\sqrt{|\dot{D}(\mu_n^\pm)|}k+\mathcal{O}(|k|^2)\tag{5.14}$$

holds true.

Consider the equation (5.1) for $\lambda=\mu_n^\pm\mp k^2$. We are looking for the solution to this equation satisfying the conditions

$$\begin{aligned}u(x,\lambda)&=\mathrm{e}^{-\varkappa_n^\pm(k)x}\Phi_{n,+}^\pm(x,k), && x\geqslant x_1,\\ u(x,\lambda)&=\mathrm{e}^{\varkappa_n^\pm(k)x}\Phi_{n,-}^\pm(x,k), && x\leqslant x_0,\end{aligned}\tag{5.15}$$

where $\Phi^{\pm}_{n,\pm}$ are 1-periodic w.r.t. x functions. In order to solve this problem we again employ the scheme borrowed from [9, Ch. XIV, Sec. 4]. The main difference is that the analogue of the function v in (5.8) is defined in a more complicated way that allows us to avoid singularities at the point $k=0$ for an analogue of the operator $\mathcal{T}_\varepsilon$ in (5.10). To define the analogue of the function v we first introduce additional notations.

We denote $\tau_n^{\pm} := \pm(-1)^n$, and

$$\begin{aligned}
\varphi^{\pm}_{n,1}(x,k) &:= \sqrt{\tau_n^{\pm}\theta_2(\lambda)}\left(\theta_1(x,\lambda)+\frac{\rho_n^{\pm}(k)-\theta_1(\lambda)}{\theta_2(\lambda)}\theta_2(x,\lambda)\right),\\
\varphi^{\pm}_{n,2}(x,k) &:= \sqrt{\tau_n^{\pm}\theta_2(\lambda)}\left(\theta_1(x,\lambda)+\frac{\big(\rho_n^{\pm}(k)\big)^{-1}-\theta_1(\lambda)}{\theta_2(\lambda)}\theta_2(x,\lambda)\right),
\end{aligned} \tag{5.16}$$

if $\theta_2(\mu_n^{\pm})\neq 0$, and

$$\begin{aligned}
\varphi^{\pm}_{n,1}(x,k) &:= \sqrt{-\tau_n^{\pm}\theta_1'(\lambda)}\left(\frac{\rho_n^{\pm}(k)-\theta_2'(\lambda)}{\theta_1'(\lambda)}\theta_1(x,\lambda)+\theta_2(x,\lambda)\right),\\
\varphi^{\pm}_{n,2}(x,k) &:= \sqrt{-\tau_n^{\pm}\theta_1'(\lambda)}\left(\frac{\big(\rho_n^{\pm}(k)\big)^{-1}-\theta_2'(\lambda)}{\theta_1'(\lambda)}\theta_1(x,\lambda)+\theta_2(x,\lambda)\right),
\end{aligned} \tag{5.17}$$

if $\theta_2(\mu_n^{\pm})=0$. Everywhere in these formulas the symbol λ indicates the sum $\mu_n^{\pm}\mp k^2$. Item 1 of Lemma 5.3 implies that the functions $\varphi^{\pm}_{n,i}$ are well-defined.

By analogy with (5.5), (5.6), (5.7) one can check that the functions $\varphi^{\pm}_{n,i}$ can be represented as

$$\varphi^{\pm}_{n,1}(x,k)=\mathrm{e}^{\varkappa_n^{\pm}(k)x}\Phi^{\pm}_{n,1}(x,k),\quad \varphi^{\pm}_{n,2}(x,k)=\mathrm{e}^{-\varkappa_n^{\pm}(k)x}\Phi^{\pm}_{n,2}(x,k), \tag{5.18}$$

where $\Phi^{\pm}_{n,i}(x,k)$ are 1-periodic w.r.t. x for even n and 1-antiperiodic for odd n. Lemma 4.1 and 1-(anti)periodicity of $\Phi^{\pm}_{n,i}$ imply that for ε small enough these functions are holomorphic w.r.t. k small enough in the norm of $C^2[\alpha_1,\alpha_2]$ for all $\alpha_1,\alpha_2\in\mathbb{R}$. Since the functions $\theta_i(x,\mu_n^{\pm})$ are real, Items 2, 3 of Lemma 5.3 imply that

$$\begin{aligned}
\operatorname{sgn}\theta_2(\mu_n^{\pm}) &= \pm(-1)^n, &&\text{if } \theta_2(\mu_n^{\pm})\neq 0,\\
-\operatorname{sgn}\theta_1'(\mu_n^{\pm}) &= \pm(-1)^n, &&\text{if } \theta_1'(\mu_n^{\pm})\neq 0.
\end{aligned}$$

Bearing in mind these relations, the holomorphy of the functions $\Phi^{\pm}_{n,i}$, (5.6), (4.2), (5.13), (5.14), one can make sure that

$$\varphi^{\pm}_{n,i}(x,0)=\phi_n^{\pm}(x),\quad i=1,2, \tag{5.19}$$

where $\phi_n^{\pm}$ are the eigenfunctions of (2.3) associated with $\mu_n^{\pm}$ and satisfying the normalization condition (2.8). The right hand side of this relation is nonzero by Item 1 of Lemma 5.3. The function $\phi_n^{\pm}$ being not identically zero, there exists a

point $x_2 \notin \overline{Q}$ such that $\phi_n^\pm(x_2) \neq 0$. Of course, the point x_2 depends on n and an edge of a lacuna. Enlarging if needed the interval (x_0, x_1) we can assume that $x_2 \in (x_0, x_1)$ and $\zeta \equiv 1$ in a neighbourhood of the point x_2.

On the functions $f \in L_2(\mathbb{R}; (x_0, x_1))$ we define the operators

$$\mathcal{G}_{n,+}^\pm(k)f := \mathcal{P}(\lambda, +\infty)f - \frac{\big(\mathcal{P}(\lambda, +\infty)f\big)(x_2, k)}{\varphi_{n,2}^\pm(x_2, k)}\varphi_{n,2}^\pm(\cdot, k),$$

$$\mathcal{G}_{n,-}^\pm(k)f := \mathcal{P}(\lambda, -\infty)f - \frac{\big(\mathcal{P}(\lambda, -\infty)f\big)(x_2, k)}{\varphi_{n,1}^\pm(x_2, k)}\varphi_{n,1}^\pm(\cdot, k),$$

where $\lambda = \mu_n^\pm \mp k^2$, and $\mathcal{P}$, we remind, is the operator in (4.3). For the brevity till the end of the section we will omit the index "$\pm$" in the notations.

By Lemma 4.1, the identity (5.19) and the assumption $\phi_n^\pm(x_2) \neq 0$ the operators $\mathcal{G}_{n,\pm}^\pm$ are holomorphic w.r.t. k small enough as the operators from $L_2(\mathbb{R}; (x_0, x_1))$ into $W_2^2(x_0, x_1)$. Let $g \in L_2(\mathbb{R}; (x_0, x_1))$ be a function. It is easy to check that the functions

$$v_+ := \mathcal{G}_{n,+}(k)g, \quad v_- := \mathcal{G}_{n,-}(k)g,$$

are solutions to the equations (5.4) for $\lambda = \mu_n^\pm \mp k^2$ vanishing at x_2. Moreover, the function v_+ satisfies the former of the relations (5.15), while v_- does the latter.

We introduce the function $v(x, k) := v_-(x, k)$, $x < x_2$, $v(x, k) := v_+(x, k)$, $x > x_2$. Let w_ε be a solution to the boundary value problem

$$\begin{gathered}\left(-\frac{d}{dx}p\frac{d}{dx} + q - \varepsilon\mathcal{L}_\varepsilon - \mathrm{i}\right)w_\varepsilon = h, \quad x \in (x_0, x_1), \quad w_\varepsilon = v, \quad x = x_0, x_1, \\ h := \begin{cases} \left(-\frac{d}{dx}p\frac{d}{dx} + q - \mathrm{i}\right)v_-, & x < x_2, \\ \left(-\frac{d}{dx}p\frac{d}{dx} + q - \mathrm{i}\right)v_+, & x > x_2. \end{cases}\end{gathered} \tag{5.20}$$

The function h belongs to $L_2(x_0, x_1)$. The problem (5.20) is uniquely solvable in $W_2^2(x_0, x_1)$ for ε small enough. Indeed, a change

$$w_\varepsilon = w_\varepsilon^{(0)} + w_\varepsilon^{(1)}, \quad w_\varepsilon^{(0)}(x) := \frac{v(x_1)(x - x_0) - v(x_0)(x - x_1)}{x_1 - x_0},$$

reduces the problem (5.20) to an equation $(\mathcal{H}_\varepsilon^{(0)} - \mathrm{i})w_\varepsilon^{(1)} = h^{(1)}$, where

$$h^{(1)} := h - \left(-\frac{d}{dx}p\frac{d}{dx} + q - \varepsilon\mathcal{L}_\varepsilon - \mathrm{i}\right)w_\varepsilon^{(0)}.$$

As it was shown above this equation is uniquely solvable. Moreover, the mapping $g \mapsto w_\varepsilon$ is a linear operator from $L_2(\mathbb{R}; (x_0, x_1))$ into $W_2^2(x_0, x_1)$ being holomorphic w.r.t. k small enough.

We construct the solution to (5.1), (5.15) as follows

$$u(x,k) := \big(1-\zeta(x)\big)v(x,k) + \zeta(x)w_\varepsilon(x,k). \tag{5.21}$$

This functions satisfies the conditions (5.15). We substitute it into the left hand side of (5.1) with $\lambda = \mu_n^\pm \mp k^2$ to obtain

$$\begin{gathered}
\left(-\frac{d}{dx}p\frac{d}{dx} + q - \varepsilon\mathcal{L}_\varepsilon - \mu_n^\pm \pm k^2\right)\big((1-\zeta)v + \zeta w_\varepsilon\big) = g + \mathcal{T}_{n,\varepsilon}^\pm(k)g,\\
\mathcal{T}_{n,\varepsilon}^\pm(k)g := -\frac{d}{dx}p(w_\varepsilon - v)\frac{d\zeta}{dx} - p\frac{d\zeta}{dx}\frac{d}{dx}(w_\varepsilon - v) + (\mathrm{i} - \mu_n^\pm \pm k^2)\zeta(w_\varepsilon - v),
\end{gathered}$$

that leads us to the equation

$$g + \mathcal{T}_{n,\varepsilon}^\pm(k)g = f. \tag{5.22}$$

The operator $\mathcal{T}_{n,\varepsilon}^\pm$ is compact in $L_2(x_0,x_1)$.

Lemma 5.4. *The problem (5.1), (5.15) is equivalent to the equation (5.22) for all k small enough. Namely, for each solution of (5.22) there exists the unique solution to (5.1), (5.15) defined by (5.21). For each solution u of (5.1), (5.15) there exists the unique function g satisfying to the equation(5.22) and related with u by (5.21).*

The proof of the lemma is completely analogous to that of Lemma 5.1. The formulas expressing $v_\pm$, w_ε, g via the solution u of (5.1), (5.15) are as follows,

$$\begin{aligned}
&v_\pm(x,k) := u(x,k) - \zeta(x)U_\pm(x,k), \qquad \pm x > \pm x_2,\\
&w_\varepsilon(x,k) := u + \big(1-\zeta(x)\big)U_\pm(x,k), \quad \pm x > \pm x_2,\\
&g(x,k) = \begin{cases} \left(-\dfrac{d}{dx}p\dfrac{d}{dx} + q - \mu_n^\pm \pm k^2\right)v_-, & x < x_2,\\ \left(-\dfrac{d}{dx}p\dfrac{d}{dx} + q - \mu_n^\pm \pm k^2\right)v_+, & x > x_2,\end{cases}
\end{aligned}$$

where $U_\pm$ are solutions to the boundary value problems:

$$\begin{gathered}
\left(-\frac{d}{dx}p\frac{d}{dx} + q - \varepsilon\mathcal{L}_\varepsilon - \mathrm{i}\right)U_- = \varepsilon\mathcal{L}_\varepsilon u, \quad x \in (x_0,x_2),\\
U_- = 0, \quad x = x_0, \quad U_- = u, \quad x = x_1,\\
\left(-\frac{d}{dx}p\frac{d}{dx} + q - \varepsilon\mathcal{L}_\varepsilon - \mathrm{i}\right)U_+ = \varepsilon\mathcal{L}_\varepsilon u, \quad x \in (x_2,x_1),\\
U_+ = u, \quad x = x_2, \quad U_+ = 0, \quad x = x_2.
\end{gathered}$$

Proof of Theorem 2.3. As it was said above, to finish the proof of Theorem 2.3 it remains to show that the edges of the non-degenerate lacunas in the spectrum

of $\mathcal{H}_0$ are not accumulation points for the eigenvalues of $\mathcal{H}_\varepsilon$, and we should also check that if such an edge is an eigenvalue, it is of finite multiplicity. We will prove these facts by analogy with the proof of Lemma 5.2.

Let λ be an eigenvalue of the operator $\mathcal{H}_\varepsilon$ lying in a vicinity of an edge $\mu_n^\pm$ of a non-degenerate lacuna. Then the corresponding eigenfunction is a nontrivial solution to the problem

$$\left(-\frac{d}{dx}p\frac{d}{dx}+q-\varepsilon\mathcal{L}_\varepsilon-\mu_n^\pm\pm k^2\right)\psi=0,\quad x\in\mathbb{R},\tag{5.23}$$

satisfying the condition (5.15), where $\operatorname{Re}\varkappa_n^\pm(k)\geqslant 0$. By (5.14) the last inequality is equivalent to $\operatorname{Re}k\geqslant 0$. Due to Lemma 5.4 it means that the corresponding equation (5.22) has a nontrivial solution. The compactness of the operator $\mathcal{T}_{n,\varepsilon}^\pm(k)$ follows that the number of such solutions is finite. Thus, if $\mu_n^\pm$ is an eigenvalue of $\mathcal{H}_\varepsilon$, it is of finite multiplicity.

Since by Lemma 5.2 the number of the eigenvalues of $\mathcal{H}_\varepsilon$ is at most countable, it follows that in any small neighbourhood of $\mu_n^\pm$ there exists a point λ_* not belonging to the spectrum of $\mathcal{H}_\varepsilon$. Let k_* be a value of the parameter k associated with λ_* and $\operatorname{Re}k_*>0$. Then the problem (5.23), (5.15) with $k=k_*$ has no nontrivial solutions that by Lemma 5.4 means the absence of the nontrivial solutions to the equation (5.22) with $k=k_*$. Thus, the operator $(\mathrm{I}+\mathcal{T}_\varepsilon(k_*))$ is boundedly invertible. Together with the holomorphy of this operator it allows us to apply Theorem 7.1 in [9, Ch. XV, Sec. 7] and to conclude that the operator $(\mathrm{I}+\mathcal{T}_\varepsilon)^{-1}$ is meromorphic in k and its poles can not accumulate at the internal point of the considered neighbourhood. Each value of the parameter k corresponding to the eigenvalue of operator $\mathcal{H}_\varepsilon$ being close to $\mu_n^\pm$ is a pole of the operator $(\mathrm{I}+\mathcal{T}_\varepsilon)^{-1}$. Thus, the eigenvalues of the operator $\mathcal{H}_\varepsilon$ lying in the vicinity of $\mu_n^\pm$ can not accumulate at $\mu_n^\pm$. ☐

6 Proof of Theorem 2.5

In the present section we prove Theorem 2.5. Moreover, we provide an example showing that under violation of the hypothesis of this theorem the operator $\mathcal{H}_\varepsilon$ can have embedded eigenvalues.

By analogy with the proof of Theorem 2.4 one can show that if an eigenvalue λ of $\mathcal{H}_\varepsilon$ is embedded, the associated eigenfunction satisfies the identity (4.6). In what follows we regard this identity as proven.

Proof of Item (1) of Theorem 2.5. We argue by contradiction. Let $\lambda\in\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon)$ be an eigenvalues of $\mathcal{H}_\varepsilon$, and ψ is an associated eigenfunction. Let x_2 be the left endpoint of the support of ψ and

$$\psi(x)\not\equiv 0,\quad x_2<x<x_2+\delta,\tag{6.1}$$

where δ is a small number. Since $\psi\in W_2^2(\mathbb{R})\subset C^1(\mathbb{R})$, the initial conditions

$$\psi(x_2)=\psi'(x_2)=0 \tag{6.2}$$

hold true. The definition (4.3) of $\mathcal{P}$ implies that the initial problem (4.5), (6.2) is equivalent to an integral equation

$$\big(\mathrm{I}-\varepsilon\mathcal{L}_\varepsilon\mathcal{P}(\lambda,x_2)\big)\psi=0. \tag{6.3}$$

In view of (4.3), Lemma 4.1 and the estimate (2.4), for any function $u\in W_2^2(x_2,x_2+\delta)$ we obtain

$$\|\mathcal{L}_\varepsilon\mathcal{P}(\lambda,x_2)u\|_{L_2(x_2,x_2+\delta)}\leqslant C\delta\|u\|_{L_2(x_2,x_2+\delta)},$$

where the constant C is independent of δ and ε. Therefore, for δ small enough the operator $\mathcal{L}_\varepsilon\mathcal{P}(\lambda,x_2)$ is a contraction operator in $L_2(x_2,x_2+\delta)$ and the equation (6.3) thus has the trivial solution only. It contradicts to (6.1). $\square$

Proof of Item (2) of Theorem 2.5. In view of Theorem 2.4 it is sufficient to show that there are no embedded eigenvalues of $\mathcal{H}_\varepsilon$ tending to infinity as $\varepsilon\to 0$.

Let $\lambda\in\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon)$ is an eigenvalue of $\mathcal{H}_\varepsilon$ greater than one, and ψ is an associated eigenfunction normalized in $L_2(\mathbb{R})$. This eigenfunction satisfies the equation

$$\left(-\frac{d}{dx}p_\varepsilon\frac{d}{dx}+q-\lambda-\varepsilon\widetilde{\mathcal{L}}_\varepsilon\right)\psi=0, \tag{6.4}$$

where $p_\varepsilon:=p+\varepsilon a_\varepsilon$. We multiply this equation by $\overline{\psi}$ and integrate by parts bearing in mind (4.6). It results in

$$\|\sqrt{p_\varepsilon}\psi'\|^2_{L_2(Q)}+(q\psi,\psi)_{L_2(Q)}+\varepsilon(\widetilde{\mathcal{L}}_\varepsilon\psi,\psi)_{L_2(Q)}=\lambda. \tag{6.5}$$

The function p_ε is positive for ε small enough since by (2.5) and the function a_ε being compactly supported we have

$$\varepsilon\max_{\overline{Q}}|a_\varepsilon(x)|=\varepsilon\max_{\overline{Q}}\left|\int\limits_{x_0}^{x}a_\varepsilon'(t)\,\mathrm{d}t\right|\leqslant\varepsilon|x_1-x_0|\max_{\overline{Q}}|a_\varepsilon'(t)|\xrightarrow[\varepsilon\to 0]{}0.$$

Bearing in mind this convergence, by (6.5), (2.1) and the uniform in ε boundedness of the operator $\widetilde{\mathcal{L}}_\varepsilon : W_2^1(Q)\to L_2(Q)$ we obtain

$$\|\psi'\|^2_{L_2(Q)}\leqslant C+\lambda+C\varepsilon\|\psi'\|^2_{L_2(Q)},\quad \|\psi\|_{W_2^1(Q)}\leqslant C\sqrt{\lambda},\quad \|\widetilde{\mathcal{L}}_\varepsilon\psi\|_{L_2(Q)}\leqslant C\sqrt{\lambda}, \tag{6.6}$$

where C are constants independent of ε and λ. We multiply the equation (6.4) by $\mathrm{e}^{\alpha x}p_\varepsilon\overline{\psi}'$, $\alpha>0$, and integrate by parts taking into account (4.6):

$$
\begin{aligned}
0=&-2\,\mathrm{Re}\int\limits_Q \mathrm{e}^{\alpha x}p_\varepsilon\overline{\psi}'(p_\varepsilon\psi')'\,\mathrm{d}x+2\,\mathrm{Re}\int\limits_Q q\mathrm{e}^{\alpha x}p_\varepsilon\psi\overline{\psi}'\,\mathrm{d}x+2\varepsilon\,\mathrm{Re}\int\limits_Q \mathrm{e}^{\alpha x}p_\varepsilon\overline{\psi}'\widetilde{\mathcal{L}}_\varepsilon\psi\,\mathrm{d}x\\
&-2\lambda\,\mathrm{Re}\int\limits_Q \mathrm{e}^{\alpha x}p_\varepsilon\overline{\psi}'\psi\,\mathrm{d}x=\alpha\|\mathrm{e}^{\frac{\alpha\cdot}{2}}p_\varepsilon\psi'\|^2_{L_2(Q)}+\lambda\big((\mathrm{e}^{\alpha x}p_\varepsilon)'\psi,\psi\big)_{L_2(Q)}\\
&+2\,\mathrm{Re}\left(q\mathrm{e}^{\alpha x}p_\varepsilon\overline{\psi}',\psi\right)_{L_2(Q)}+2\varepsilon\,\mathrm{Re}\left(\mathrm{e}^{\alpha x}p_\varepsilon\widetilde{\mathcal{L}}_\varepsilon\psi,\psi'\right)_{L_2(Q)}.
\end{aligned}
\tag{6.7}
$$

Since $(\mathrm{e}^{\alpha x}p_\varepsilon)'=\mathrm{e}^{\alpha x}(\alpha p_\varepsilon+p_\varepsilon')$, by (2.1) and (2.5) we infer that there exists a constant α independent of ε and λ such that

$$
\big(\mathrm{e}^{\alpha x}(p+\varepsilon a_\varepsilon)\big)'\geqslant 1,\quad x\in\overline{Q}.
$$

We substitute this estimate and (6.6) into (6.7) and get

$$
\lambda\leqslant 2\left|\left(q\mathrm{e}^{\alpha x}(p+\varepsilon a_\varepsilon)\overline{\psi}',\psi\right)_{L_2(Q)}\right|+2\varepsilon\left|\left(\mathrm{e}^{\alpha x}(p+\varepsilon a_\varepsilon)\widetilde{\mathcal{L}}_\varepsilon\psi,\psi'\right)_{L_2(Q)}\right|\leqslant C\sqrt{\lambda}+\varepsilon C\lambda,
$$

where the constant C is independent of ε and λ. Therefore, $\lambda\leqslant C\sqrt{\lambda}$, that implies $\lambda\leqslant C$, where the constant C is independent of ε and λ. ☐

In the remaining part of the section we give the example showing that under violation of the hypothesis of this theorem the operator $\mathcal{H}_\varepsilon$ can have an eigenvalue embedded into the continuous spectrum.

We set $p\equiv 1$, $q\equiv 0$, i.e., $\mathcal{H}_0=-\frac{d^2}{dx^2}$. We introduce the operator $\mathcal{L}_\varepsilon$ as follows,

$$
\begin{aligned}
&(\mathcal{L}_\varepsilon u)(x):=2\xi_\varepsilon(x)l_\varepsilon u, && l_\varepsilon u:=\varepsilon^{-1}\big(u'(\varepsilon^\alpha)-u'(0)\big), && \alpha\geqslant 2,\\
&\xi_\varepsilon(x):=c_\varepsilon\chi_\varepsilon(x)\sin\left(\nu_\varepsilon|x|\right), && c_\varepsilon:=\left(\frac{2\pi[\nu_\varepsilon]}{\nu_\varepsilon}-\varepsilon^\alpha\right)^{-1}, && \nu_\varepsilon:=\frac{\pi}{2\varepsilon^\alpha},
\end{aligned}
$$

where χ_ε is the characteristic function of the interval $\left(-\frac{2\pi[\nu_\varepsilon]}{\nu_\varepsilon},\frac{2\pi[\nu_\varepsilon]}{\nu_\varepsilon}\right)$, $[\nu_\varepsilon]$ is the integer part of ν_ε. Let us check that the operator $\mathcal{L}_\varepsilon$ satisfies all needed requirements with $Q=(-2\pi,2\pi)$. It is clear that due to embedding $W_2^2(Q)\subset C^1(Q)$ the functional l_ε is a linear functional on the space $W_2^2(Q)$, and $\mathcal{L}_\varepsilon$ is a linear operator from $W_2^2(Q)$ into $L_2(\mathbb{R};Q)$. By the estimate

$$
|l_\varepsilon u|\leqslant\varepsilon^{-1}\int\limits_0^{\varepsilon^\alpha}|u''(t)|\,\mathrm{d}t\leqslant\varepsilon^{\alpha/2-1}\|u''\|_{L_2(0,\varepsilon)}\leqslant C\|u\|_{W_2^2(Q)},
$$

where the constant C is independent of ε, we obtain the uniform in ε boundedness for the functional l_ε on $W_2^2(Q)$. It yields the same property for $\mathcal{L}_\varepsilon$.

Let us prove now that $\lambda_\varepsilon := \nu_\varepsilon^2$ is an eigenvalue of the operator $\mathcal{H}_\varepsilon$, and

$$\psi_\varepsilon(x) := -\frac{\varepsilon}{\nu_\varepsilon}\int\limits_Q \sin\left(\nu_\varepsilon|x-t|\right)\xi_\varepsilon(t)\,\mathrm{d}t$$

is an associated eigenfunction. Clearly, $\psi_\varepsilon \in W_{2,loc}^2(\mathbb{R})$. Employing the formula

$$\psi_\varepsilon'(x) = -\varepsilon\int\limits_Q \cos\left(\nu_\varepsilon(x-t)\right)\mathrm{sgn}(x-t)\xi_\varepsilon(t)\,\mathrm{d}t, \quad x\in Q,$$

by direct calculations we check that $\psi_\varepsilon'(0)=0$, $\psi_\varepsilon'(\varepsilon^\alpha)=\varepsilon$, $l_\varepsilon\psi_\varepsilon=1$. This equality and the definition of the function ψ_ε imply that this function is a solution to (4.5) for $\lambda=\lambda_\varepsilon$. The definition of ψ_ε also implies that $\psi_\varepsilon(x)\equiv 0$ for $x\notin Q$, and thus $\psi_\varepsilon \in W_2^2(\mathbb{R})$. Therefore, λ_ε is an eigenvalue of $\mathcal{H}_\varepsilon$. Since $\sigma_{\mathrm{c}}(\mathcal{H}_0)=[0,+\infty)$, by Theorem 2.2 we have an identity $\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon)=[0,+\infty)$. The eigenvalue λ_ε is positive, and thus $\lambda_\varepsilon\in\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon)$.

7 Auxiliary statements

In the present section we prove certain auxiliary statements needed in proof of Theorems 2.7, 2.9.

Throughout the section by $\mu_n^\pm$ we mean an edge of a non-degenerate lacuna in the spectrum of $\mathcal{H}_0$.

The identity (4.2) and the definition (5.16), (5.17) of the functions $\varphi_{n,i}^\pm$ imply that

$$W[\varphi_{n,1}^\pm,\varphi_{n,2}^\pm] = \frac{\tau_n^\pm}{p(x)}\left(\frac{1}{\rho_n^\pm(k)}-\rho_n^\pm(k)\right), \tag{7.1}$$

where, we remind, the functions $\rho_n^\pm$ were introduced in (5.12). By $\mathcal{G}_n^\pm(k)$ we denote an integral operator defined in $L_2(Q)$:

$$(\mathcal{G}_n^\pm(k)f)(x) := \int\limits_Q G_n^\pm(x,t,k)f(t)\,\mathrm{d}t,$$

$$G_n^\pm(x,t,k) := \frac{\tau_n^\pm}{\rho_n^\pm(k)-\left(\rho_n^\pm(k)\right)^{-1}}\begin{cases}\varphi_{n,1}^\pm(x,k)\varphi_{n,2}^\pm(t,k), & t>x,\\ \varphi_{n,1}^\pm(t,k)\varphi_{n,2}^\pm(x,k), & t<x.\end{cases}$$

Lemma 7.1. *Let $k\in\mathbb{C}$ be small enough. Then*

(1). For any function $f\in L_2(\mathbb{R};Q)$ and $k\neq 0$ the solution to the equation

$$\left(-\frac{d}{dx}p\frac{d}{dx}+q-\mu_n^\pm \pm k^2\right)u=f, \quad x\in\mathbb{R}, \tag{7.2}$$

satisfying the conditions (5.15) is given by $u(x,k) = (\mathcal{G}_n^\pm(k)f)(x) \in W_{2,loc}^2(\mathbb{R})$. *For* $x \notin Q$ *the function* u *is of the form*

$$u(x,k) = \frac{\tau_n^\pm \mathrm{e}^{-\varkappa_n^\pm(k)x}\Phi_{n,2}^\pm(x,k)}{\rho_n^\pm(k) - \big(\rho_n^\pm(k)\big)^{-1}} \int\limits_Q \varphi_{n,1}^\pm(t,k)f(t)\,\mathrm{d}t, \tag{7.3}$$

if x *lies to right w.r.t.* Q, *and*

$$u(x,k) = \frac{\tau_n^\pm \mathrm{e}^{\varkappa_n^\pm(k)x}\Phi_{n,1}^\pm(x,k)}{\rho_n^\pm(k) - \big(\rho_n^\pm(k)\big)^{-1}} \int\limits_Q \varphi_{n,2}^\pm(t,k)f(t)\,\mathrm{d}t, \tag{7.4}$$

if x *lies to the left w.r.t.* Q.

(2). For any $\alpha_1,\alpha_2 \in \mathbb{R}$ *the operator* $\mathcal{G}_n^\pm(k) : L_2(Q) \to W_2^2(\alpha_1,\alpha_2)$ *is boundedly meromorphic w.r.t.* k. *The representation*

$$\mathcal{G}_n^\pm(k) = k^{-1}\mathcal{G}_{n,-1}^\pm + \mathcal{G}_{n,0}^\pm + k\mathcal{G}_{n,1}(k),$$

holds true, where the operator $\mathcal{G}_{n,-1}^\pm$ *is defined as*

$$(\mathcal{G}_{n,-1}^\pm f)(x) = \pm\frac{(f,\phi_n^\pm)_{L_2(Q)}}{2\sqrt{|\dot{D}(\mu_n^\pm)|}}\phi_n^\pm(x),$$

while the operator $\mathcal{G}_{n,1}^\pm : L_2(Q) \to W_2^2(\alpha_1,\alpha_2)$ *is bounded and holomorphic in* k.

Remark 7.1. We remind that the functions $\Phi_{n,i}^\pm$ in (7.3), (7.4) in Item (1) of the lemma were defined by (5.18), the operator $\mathcal{G}_{n,0}^\pm$ was defined by (2.9), and $\phi_n^\pm$ are the eigenfunctions of (2.3) normalized by (2.8).

Item (1) of the lemma follows directly from the definition of the function $G_n^\pm$ and the identities (7.1), (5.18). The validity of Item (2) is due to the definition of $G_n^\pm$, holomorphy in k of $\varphi_{n,i}^\pm$, $\Phi_{n,i}^\pm$ and $\rho_n^\pm$, and the identity (5.13).

The uniform boundedness of the operator $\mathcal{L}_\varepsilon$ and Item (2) of Lemma 7.1 follows that for all ε and $k \in \mathbb{C}$ small enough the operator $\mathcal{L}_\varepsilon\big(\mathcal{G}_{n,0}^\pm + k\mathcal{G}_{n,1}^\pm(k)\big) : L_2(Q) \to L_2(Q)$ is bounded uniformly in ε and k. Thus, for all ε and k small enough the bounded operator

$$\mathcal{A}_n^\pm(\varepsilon,k) := \Big(\mathrm{I} - \varepsilon\mathcal{L}_\varepsilon\big(\mathcal{G}_{n,0}^\pm + k\mathcal{G}_{n,1}^\pm(k)\big)\Big)^{-1}$$

is well-defined in $L_2(Q)$. Item (2) of Lemma 7.1 follows that for all ε and k small enough the operator $\mathcal{A}_n^\pm(\varepsilon,k)$ is boundedly holomorphic w.r.t. k, and an uniform in k convergence

$$\mathcal{A}_n^\pm(\varepsilon,k) \xrightarrow[\varepsilon\to 0]{} \mathrm{I} \tag{7.5}$$

holds true.

Lemma 7.2. *For all ε and k small enough the equation*

$$k \mp \frac{\varepsilon}{2\sqrt{|\dot{D}(\mu_n^\pm)|}} \big(\mathcal{A}_n^\pm(\varepsilon,k)\mathcal{L}_\varepsilon \phi_n^\pm, \phi_n^\pm\big)_{L_2(Q)} = 0 \tag{7.6}$$

has the unique solution $k_{\varepsilon,n}^\pm$. The asymptotic formulas

$$k_{\varepsilon,n}^\pm = \pm \frac{\varepsilon}{2\sqrt{|\dot{D}(\mu_n^\pm)|}} \big(\mathcal{A}_n^\pm(\varepsilon,0)\mathcal{L}_\varepsilon \phi_n^\pm, \phi_n^\pm\big)_{L_2(Q)} (1+\mathcal{O}(\varepsilon^2)), \tag{7.7}$$

$$k_{\varepsilon,n}^\pm = \varepsilon\big(k_{n,\varepsilon}^{\pm,1} + \varepsilon k_{n,\varepsilon}^{\pm,2}\big) + \mathcal{O}(\varepsilon^3), \tag{7.8}$$

holds true, where $k_{n,\varepsilon}^{\pm,i}$ is from (2.13).

Proof. Since the operator $\mathcal{A}_n^\pm(\varepsilon,k)$ is holomorphic in k, the function $k \mapsto \big(\phi_n^\pm, \mathcal{A}_n^\pm(\varepsilon,k)\mathcal{L}_\varepsilon\phi_n^\pm\big)_{L_2(Q)}$ is holomorphic in k for each value of ε. Moreover, by (7.5) and the uniform boundedness of $\mathcal{L}_\varepsilon$ this function is bounded uniformly in ε and k. Let δ be a small number. Then for ε small enough and $|k| = \delta$ we have the estimate

$$\varepsilon \left| \frac{\pm 1}{2\sqrt{|\dot{D}(\mu_n^\pm)|}} \big(\mathcal{A}_n^\pm(\varepsilon,k)\mathcal{L}_\varepsilon \phi_n^\pm, \phi_n^\pm\big)_{L_2(Q)} \right| < |k|.$$

By Rouche theorem it follows that the function

$$k \mapsto k \mp \frac{\varepsilon}{2\sqrt{|\dot{D}(\mu_n^\pm)|}} \big(\mathcal{A}_n^\pm(\varepsilon,k)\mathcal{L}_\varepsilon \phi_n^\pm, \phi_n^\pm\big)_{L_2(Q)}$$

has the same amount of zeros inside the disk $|k| \leqslant \delta$ as the function $k \mapsto k$ does, i.e., exactly one zero. Thus, for ε small enough the equation (7.6) has the unique root inside the disk $|k| \leqslant \delta$ which we indicate as $k_{\varepsilon,n}^\pm$.

An obvious identity

$$\frac{\partial \mathcal{A}_n^\pm}{\partial k}\Big(\mathrm{I} - \varepsilon\mathcal{L}_\varepsilon\big(\mathcal{G}_{n,0}^\pm + k\mathcal{G}_{n,1}^\pm\big)\Big) - \varepsilon\mathcal{A}_n^\pm\mathcal{L}_\varepsilon\left(\mathcal{G}_{n,0}^\pm + k\frac{\mathcal{G}_{n,1}^\pm}{\partial k}\right) = 0,$$

implies

$$\frac{\partial \mathcal{A}_n^\pm}{\partial k} = \varepsilon\mathcal{A}_n^\pm\mathcal{L}_\varepsilon\left(\mathcal{G}_{n,0}^\pm + k\frac{\mathcal{G}_{n,1}^\pm}{\partial k}\right)\mathcal{A}_n^\pm.$$

This formula, the convergence (7.5) and Item (2) of Lemma 7.1 give rise to the uniform in ε and k estimate

$$\left\|\frac{\partial \mathcal{A}_n^\pm}{\partial k}\right\| \leqslant C\varepsilon.$$

Employing this estimate and the formula

$$\mathcal{A}_n^\pm(\varepsilon,k) - \mathcal{A}_n^\pm(\varepsilon,0) = \int\limits_0^k \frac{\partial\mathcal{A}_n^\pm}{\partial k}(\varepsilon,z)\,\mathrm{d}z,$$

we obtain that

$$\mathcal{A}_n^{\pm}(\varepsilon,k)=\mathcal{A}_n^{\pm}(\varepsilon,0)+\mathcal{O}(\varepsilon|k|).$$

We substitute this identity into (7.6) to get

$$\begin{gathered}
k_{\varepsilon,n}^{\pm}\mp\frac{\varepsilon}{2\sqrt{|\dot{D}(\mu_n^{\pm})|}}\big(\mathcal{A}_n^{\pm}(\varepsilon,0)\mathcal{L}_\varepsilon\phi_n^{\pm},\phi_n^{\pm}\big)_{L_2(Q)}+\mathcal{O}(\varepsilon^2|k_{\varepsilon,n}^{\pm}|)=0,\\
k_{\varepsilon,n}^{\pm}\big(1+\mathcal{O}(\varepsilon^2)\big)=\pm\frac{\varepsilon}{2\sqrt{|\dot{D}(\mu_n^{\pm})|}}\big(\mathcal{A}_n^{\pm}(\varepsilon,0)\mathcal{L}_\varepsilon\phi_n^{\pm},\phi_n^{\pm}\big)_{L_2(Q)}.
\end{gathered}$$

The last relation implies the asymptotics (7.7).

Due to the definition of $\mathcal{A}_n^{\pm}(\varepsilon,k)$ the identity

$$\mathcal{A}_n^{\pm}(\varepsilon,k)=\mathrm{I}+\varepsilon\mathcal{L}_\varepsilon\mathcal{G}_{n,0}^{\pm}+\mathcal{O}(\varepsilon^2+\varepsilon|k|) \tag{7.9}$$

holds true. We substitute this identity into the asymptotic (7.7) and deduce that

$$k_{\varepsilon,n}^{\pm}=\varepsilon\big(k_{n,\varepsilon}^{\pm,1}+\varepsilon k_{n,\varepsilon}^{\pm,2}\big)+\mathcal{O}\big(\varepsilon^3+\varepsilon^2|k_{\varepsilon,n}^{\pm}|\big).$$

Now it sufficient to employ the estimate $k_{\varepsilon,n}^{\pm}=\mathcal{O}(\varepsilon)$ yielded by (7.7) to complete the proof. ☐

8 Proof of Theorems 2.7, 2.9

Proof of Theorem 2.7. In the proof we employ the approach suggested in [10] which is a modification of Birman-Schwinger principle. Let $\lambda=\mu_n^{\pm}\mp k^2$ be an eigenvalue of $\mathcal{H}_\varepsilon$ lying in a vicinity of $\mu_n^{\pm}$. Then an associated eigenfunction ψ satisfies the equation (7.2) with $f=\varepsilon\mathcal{L}_\varepsilon\psi$ as well as to the conditions (5.15), where $\operatorname{Re}\varkappa_n^{\pm}(k)>0$. By (5.14) the last inequality is equivalent to $\operatorname{Re}k>0$. By Item (1) of Lemma 7.1 we conclude that $\psi=\mathcal{G}_n^{\pm}(k)f$. We apply the operator $\varepsilon\mathcal{L}_\varepsilon$ to this relation to obtain the equation for the function f,

$$f-\varepsilon\mathcal{L}_\varepsilon\mathcal{G}_n^{\pm}(k)f=0. \tag{8.1}$$

The operator $\mathcal{L}_\varepsilon\mathcal{G}_n^{\pm}(k)$, as it follows from Item (1) of Lemma 7.1 and uniform boundedness of the operator $\mathcal{L}_\varepsilon$, is a bounded operator in $L_2(Q)$. The equation (8.1) can be hence considered as an equation in this space.

If f is a nontrivial solution to (8.1) for some k, it follows that the function $\psi:=\mathcal{G}_n^{\pm}(k)f$ is a nontrivial solution to

$$\left(-\frac{d}{dx}p\frac{d}{dx}+q-\varepsilon\mathcal{L}_\varepsilon-\mu_n^{\pm}\pm k^2\right)\psi=0,\quad x\in\mathbb{R}, \tag{8.2}$$

behaving at infinity in accordance with (7.3), (7.4). This function is an element of $W_2^2(\mathbb{R})$, if and only if $\operatorname{Re}k>0$. Thus, the number $\lambda=\mu_n^{\pm}\mp k^2$ is an eigenvalue of

$\mathcal{H}_\varepsilon$, if and only if $\operatorname{Re} k > 0$. Therefore, the problem on eigenvalues of $\mathcal{H}_\varepsilon$ tending to $\mu_n^\pm$ as $\varepsilon \to 0$ reduces to the problem on finding the values of k with $\operatorname{Re} k > 0$ for which the equation (8.1) has a nontrivial solution.

In accordance with Item (2) of Lemma 7.1, the equation (8.1) can be rewritten as

$$f - \frac{\varepsilon}{k}\mathcal{L}_\varepsilon \mathcal{G}_{n,-1}^\pm f - \varepsilon \mathcal{L}_\varepsilon\big(\mathcal{G}_{n,0}^\pm + k\mathcal{G}_{n,1}(k)\big) f = 0.$$

We apply the operator $\mathcal{A}_n^\pm(\varepsilon, k)$ to this equation and obtain

$$f \mp \frac{\varepsilon (f, \phi_n^\pm)_{L_2(Q)}}{2k\sqrt{|\dot{D}(\mu_n^\pm)|}} \mathcal{A}_n^\pm(\varepsilon, k)\mathcal{L}_\varepsilon \phi_n^\pm = 0. \tag{8.3}$$

Let f be a nontrivial solution to (8.1). In this case $(f, \phi_n^\pm)_{L_2(Q)} \neq 0$, since otherwise by (8.3) we have $f \equiv 0$. Taking this fact into account, we calculate the inner product of equation (8.3) with $\phi_n^\pm$ in $L_2(Q)$ and arrive at the equation. Hence, the values of the parameter k for those the equation (8.2) has a nontrivial solution are the roots of the equation (7.6). For $k = k_{\varepsilon,n}^\pm$ the equation (8.3), and, therefore, (8.1) has a solution

$$f_{\varepsilon,n}^\pm = \varepsilon \mathcal{A}_n^\pm(\varepsilon, k_{\varepsilon,n}^\pm)\mathcal{L}_\varepsilon \phi_n^\pm.$$

This fact can be checked easily by substituting $f_{\varepsilon,n}^\pm$ into (8.3) and bearing in mind (7.6). The corresponding solution of the equation (8.2) is given by

$$\psi_{\varepsilon,n}^\pm = \varepsilon \mathcal{G}_n^\pm\big(k_{\varepsilon,n}^\pm\big)\mathcal{A}_n^\pm\big(\varepsilon, k_{\varepsilon,n}^\pm\big)\mathcal{L}_\varepsilon \phi_n^\pm. \tag{8.4}$$

This solution is nontrivial since by Item (2) of Lemma 7.1, (7.9), (7.6) and uniform in ε boundedness of $\mathcal{L}_\varepsilon$ for any $\alpha_1, \alpha_2 \in \mathbb{R}$ in the norm of $W_2^2(\alpha_1, \alpha_2)$ the identity (2.14) is valid. Therefore, $f_{\varepsilon,n}^\pm \not\equiv 0$. The relation (8.3) implies that all the solutions to (8.1) with $k = k_{\varepsilon,n}^\pm$ are proportional to $f_{\varepsilon,n}^\pm$, and this is why the solution $f_{\varepsilon,n}^\pm$ is unique up to a multiplicative constant. Thus for ε small enough there exists the unique k in a small neighbourhood of zero for which the equation (8.1) has a nontrivial solution. This value of k is a root of the equation (7.6). The function $\psi_{\varepsilon,n}^\pm$ is an eigenfunction of the operator $\mathcal{H}_\varepsilon$ only in the case $\operatorname{Re} k_{\varepsilon,n}^\pm > 0$. Otherwise the operator $\mathcal{H}_\varepsilon$ has no eigenvalues converging to $\mu_n^\pm$ as $\varepsilon \to 0$. Therefore, the operator $\mathcal{H}_\varepsilon$ has at most one eigenvalue which converges to $\mu_n^\pm$. If exists, it is simple and is given by $\lambda_{\varepsilon,n}^\pm = \mu_n^\pm \mp \big(k_{\varepsilon,n}^\pm\big)^2$. By (7.7) we obtain that the inequality $\operatorname{Re} k_{\varepsilon,n}^\pm > 0$ is equivalent to (2.10). If this inequality holds true, the asymptotic expansions for $\lambda_{\varepsilon,n}^\pm$ follows immediately from (7.7), (7.8). The associated eigenfunction satisfies the expansion (2.14). □

Proof of Theorem 2.9. As it was established in the proof of Theorem 2.7, the criterion of the existence of the eigenvalue is the inequality $\operatorname{Re} k_{\varepsilon,n}^\pm > 0$. The asymptotic (7.8) implies that the sufficient condition this inequality to be true is the estimate (2.15), while the sufficient condition of violation is the estimate (2.16). □

In conclusion let us find out the behaviour of $\psi_{\varepsilon,n}^{\pm}$ at infinity. This function satisfies (7.3), (7.4), and hence for x lying to the right w.r.t. Q we have the relation

$$\begin{aligned}
&\psi_{\varepsilon,n}^{\pm}(x) = c(\varepsilon)\mathrm{e}^{-\varkappa_n^{\pm}(k_{\varepsilon,n}^{\pm})}\Phi_{n,2}^{\pm}(x,k_{\varepsilon,n}^{\pm}),\\
&c(\varepsilon) = \frac{\tau_n^{\pm}}{\rho_n^{\pm}(k_{\varepsilon,n}^{\pm}) - \big(\rho_n^{\pm}(k_{\varepsilon,n}^{\pm})\big)^{-1}}\int\limits_{\mathbb{R}} \varphi_{n,1}^{\pm}(t,k_{\varepsilon,n}^{\pm})g_{\varepsilon,n}^{\pm}(t)\,\mathrm{d}t.
\end{aligned}$$

The identity $\varphi_{n,1}^{\pm}(\cdot,k_{\varepsilon,n}^{\pm}) = \phi_n^{\pm}(\cdot) + \mathcal{O}(|k_{\varepsilon,n}^{\pm}|)$, the asymptotics (5.13) and the equation (7.6) yield

$$\begin{aligned}
c(\varepsilon) &= \pm\frac{\varepsilon}{2\sqrt{|\dot{D}(\mu_n^{\pm})|}k_{\varepsilon,n}^{\pm}\big(1+\mathcal{O}\big(|k_{\varepsilon,n}|\big)\big)}\Big(\big(\phi_n^{\pm},\mathcal{A}_n^{\pm}(\varepsilon,k_{\varepsilon,n}^{\pm})\mathcal{L}_\varepsilon\phi_n^{\pm}\big)_{L_2(Q)} + \mathcal{O}\big(|k_{\varepsilon,n}^{\pm}|\big)\Big) =\\
&= \left(\pm\frac{\varepsilon}{2\sqrt{|\dot{D}(\mu_n^{\pm})|}k_{\varepsilon,n}^{\pm}}\big(\phi_n^{\pm},\mathcal{A}_n^{\pm}(\varepsilon,k_{\varepsilon,n}^{\pm})\mathcal{L}_\varepsilon\phi_n^{\pm}\big)_{L_2(Q)} + \mathcal{O}\big(\varepsilon|k_{\varepsilon,n}^{\pm}|\big)\right)\big(1+\mathcal{O}\big(|k_{\varepsilon,n}^{\pm}|\big)\big) =\\
&= 1+\mathcal{O}\big(|k_{\varepsilon,n}^{\pm}|\big).
\end{aligned}$$

Thus,

$$\psi_{\varepsilon,n}^{\pm}(x) = \big(1+\mathcal{O}(|k_{n,\varepsilon}^{\pm}|)\big)\mathrm{e}^{-\varkappa_n^{\pm}(k_{n,\varepsilon}^{\pm})x}\Phi_{n,2}^{\pm}(x,k_{n,\varepsilon}^{\pm}), \tag{8.5}$$

if x lies to the right w.r.t. Q. By analogy one can prove that

$$\psi_{\varepsilon,n}^{\pm}(x) = \big(1+\mathcal{O}(|k_{n,\varepsilon}^{\pm}|)\big)\mathrm{e}^{\varkappa_n^{\pm}(k_{n,\varepsilon}^{\pm})x}\Phi_{n,1}^{\pm}(x,k_{n,\varepsilon}^{\pm}), \tag{8.6}$$

if x lies to the left w.r.t. Q.

9 Examples

In this section we give some examples of the operator $\mathcal{L}_\varepsilon$. Throughout the section the symbol Q indicates certain fixed finite interval.

1. Second order differential operator. Let

$$\mathcal{L}_\varepsilon := b_2(x,\varepsilon)\frac{d^2}{dx^2} + b_1(x,\varepsilon)\frac{d}{dx} + b_0(x,\varepsilon),$$

where $b_i \in L_\infty(Q)$ are complex-valued functions so that $\operatorname{supp} b_i(\cdot,\varepsilon) \subseteq Q$, and the norms $\|b_i(\cdot,\varepsilon)\|_{L_\infty(Q)}$ are bounded uniformly in ε. The operator $\mathcal{L}_\varepsilon$ satisfies the estimate (2.4), and hence by Theorem 2.5 the continuous spectrum of $\mathcal{H}_\varepsilon$ contains no embedded eigenvalues. The primitives of the functions b_i, q, p, and the function p are continuous and have bounded variation (see, for instance, [8, Ch. VI, Sec. 1,2]). Employing this fact and applying the uniqueness theorem for Cauchy problem from [11, Ch. 1, Sec. 3, Item 3], one can check easily that each eigenvalue of $\mathcal{H}_\varepsilon$ is simple. The existence and the asymptotics of the eigenvalues converging to the edges of the non-degenerate lacunas are described by Theorems 2.7, 2.9.

In the particular case $b_1 = b_2 \equiv 0$ the coefficient $k^{\pm,1}_{\varepsilon,n}$ reads as follows

$$k^{\pm,1}_{n,\varepsilon} = \pm \frac{(\phi^{\pm}_n, b_0\phi^{\pm}_n)_{L_2(Q)}}{2\sqrt{|\dot{D}(\mu^{\pm}_n)|}}. \tag{9.1}$$

If b_0 is independent of ε and $b_0 \not\equiv 0$, $b_0 \geqslant 0$, it follows that this coefficient is independent of ε and $k^{+,1}_{n,\varepsilon} > 0$, $k^{-,1}_{n,\varepsilon} < 0$. In this case we can employ Theorem 2.9, where the estimate (2.15) is valid for $k^{+,1}_{n,\varepsilon} + \varepsilon k^{+,2}_{n,\varepsilon}$ with $C(\varepsilon) = \varepsilon^{-1}$, and the estimate (2.16) does for $k^{-,1}_{n,\varepsilon} + \varepsilon k^{-,2}_{n,\varepsilon}$ with $C(\varepsilon) = \varepsilon^{-1}$. Therefore, the eigenvalues exist only near the right edges of non-degenerate lacunas and their asymptotics are due to (2.12). If $b_0 \not\equiv 0$ and $b_0 \leqslant 0$, we have $k^{+,1}_{n,\varepsilon} < 0$, $k^{-,1}_{n,\varepsilon} > 0$, and in this case the eigenvalues exist only near the right edges of the non-degenerate lacunas. By (2.12) and (9.1) in this particular case the asymptotics of these eigenvalues are as follows,

$$\lambda^{\pm}_{\varepsilon,n} = \mu^{\pm}_n \mp \varepsilon^2 \frac{(\phi^{\pm}_n, b_0\phi^{\pm}_n)^2_{L_2(Q)}}{4|\dot{D}(\mu^{\pm}_n)|} + \mathcal{O}(\varepsilon^3).$$

2. Integral operator. Let

$$(\mathcal{L}_\varepsilon u)(x) := \int\limits_Q L_\varepsilon(x,y)u(y)dy,$$

where $\operatorname{supp} L_\varepsilon(\cdot, y) \subseteq Q$ and an uniform in ε estimate

$$\int\limits_{Q\times Q} |L_\varepsilon(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant C$$

is valid. The operator $\mathcal{L}_\varepsilon$ is bounded uniformly in ε as an operator in $L_2(Q)$, and the statement of Item (2) of Theorem 2.5 with $a_\varepsilon \equiv 0$ thus holds true for this operator. Therefore, the continuous spectrum of $\mathcal{H}_\varepsilon$ does not contain embedded eigenvalues. The coefficient $k^{\pm,1}_{n,\varepsilon}$ in (2.12) is determined by the formula

$$k^{\pm,1}_{n,\varepsilon} = \pm \int\limits_{Q\times Q} L_\varepsilon(x,y)\phi^{\pm}_n(x)\phi^{\pm}_n(y)\,\mathrm{d}x\,\mathrm{d}y.$$

In particular, if $L_\varepsilon(x,y) = \beta\overline{b(x)}b(y)$, $\operatorname{supp} b \subseteq Q$, $b \in L_2(Q)$, and $\beta \in \mathbb{C}$ is a constant, it follows that

$$k^{\pm,1}_{n,\varepsilon} = \pm \frac{\beta\left|(b,\phi^{\pm}_n)_{L_2(Q)}\right|^2}{2\sqrt{|\dot{D}(\mu^{\pm}_n)|}}, \quad k^{\pm,2}_{n,\varepsilon} = k^{\pm,1}_{n,\varepsilon} \int\limits_{Q\times Q} G^{\pm}_n(x,y)\overline{b(x)}b(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{9.2}$$

In the case $(b,\phi^{\pm}_n)_{L_2(Q)} \neq 0$ the coefficients $k^{\pm,1}_{n,\varepsilon}$ are independent of ε, and Theorem 2.9 with $C(\varepsilon) = \varepsilon^{-1}$ is applicable. In accordance with this theorem, the

eigenvalues exist near the right edges of the non-degenerate lacunas in the case $\operatorname{Re}\beta > 0$, and near the left edges if $\operatorname{Re}\beta < 0$. The asymptotics expansion for these eigenvalues are given by (2.12) and due to (9.2) read as follows:

$$\lambda^{\pm}_{\varepsilon,n} = \mu^{\pm}_n \mp \varepsilon^2 \frac{\beta^2 \left|(b,\phi^{\pm}_n)_{L_2(Q)}\right|^4}{4|\dot{D}(\mu^{\pm}_n)|} \left(1 + 2\varepsilon \int\limits_{Q\times Q} G^{\pm}_n(x,y)\overline{b(x)}b(y)\,\mathrm{d}x\,\mathrm{d}y\right) + \mathcal{O}(\varepsilon^4).$$

3. Linear functional. Let

$$\mathcal{L}_\varepsilon u := b_\varepsilon l_\varepsilon u,$$

where b_ε is a complex-valued function such that $\operatorname{supp} b_\varepsilon \subseteq Q$, and the norm $\|b_\varepsilon\|_{L_2(Q)}$ is bounded uniformly in ε. The symbol l_ε indicates the functional from $W_2^2(Q)$ in $\mathbb{C}$ bounded uniformly in ε. In this case the operator $\mathcal{L}_\varepsilon$ defined above satisfy all the requirements. As the example in the fifth section shows, there exists a function b_ε and a functional l_ε, for which the operator $\mathcal{H}_\varepsilon$ has embedded eigenvalues. At the same time, if l_ε is a functional from $W_2^1(Q)$ into $\mathbb{C}$ bounded uniformly in ε, it follows that the operator $\mathcal{L}_\varepsilon$ satisfies the hypothesis of Item (2) of Theorem 2.5 with $a_\varepsilon = 0$, and in this case the operator $\mathcal{H}_\varepsilon$ has no embedded eigenvalues.

The operator $\mathcal{L}_\varepsilon$ in this example is finite-dimensional that allows to find the function $\mathcal{A}^{\pm}_n(\varepsilon,0)\mathcal{L}_\varepsilon\phi^{\pm}_n$ explicitly:

$$\mathcal{A}^{\pm}_n(\varepsilon,0)\mathcal{L}_\varepsilon\phi^{\pm}_n = \frac{b_\varepsilon l_\varepsilon \phi^{\pm}_n}{1 - \varepsilon l_\varepsilon \mathcal{G}^{\pm}_{n,0} b_\varepsilon}.$$

It follows that

$$\left(\phi^{\pm}_n, \mathcal{A}^{\pm}_n(\varepsilon,0)\mathcal{L}_\varepsilon\phi^{\pm}_n\right)_{L_2(Q)} = \frac{(b_\varepsilon,\phi^{\pm}_n)_{L_2(Q)} l_\varepsilon \phi^{\pm}_n}{1 - \varepsilon l_\varepsilon \mathcal{G}^{\pm}_{n,0} b_\varepsilon}. \tag{9.3}$$

Now by Theorem 2.7 we obtain that the eigenvalue $\lambda^{\pm}_{\varepsilon,n}$ exists if and only if $\pm\operatorname{Re}(b_\varepsilon,\phi^{\pm}_n)_{L_2(Q)} l_\varepsilon \phi^{\pm}_n > 0$. If exists, the asymptotics for the eigenvalue $\lambda^{\pm}_{\varepsilon,n}$ is determined by the identities (2.11) and (9.3):

$$\begin{aligned}
\lambda^{\pm}_{\varepsilon,n} =& \mu^{\pm}_n \mp \frac{\varepsilon^2}{4|\dot{D}(\mu^{\pm}_n)|} \frac{\left((b_\varepsilon,\phi^{\pm}_n)_{L_2(Q)} l_\varepsilon\phi^{\pm}_n\right)^2}{(1-\varepsilon l_\varepsilon \mathcal{G}^{\pm}_{n,0} b_\varepsilon)^2}\left(1 + \mathcal{O}(\varepsilon^2)\right),\\
\lambda^{\pm}_{\varepsilon,n} =& \mu^{\pm}_n \mp \frac{\varepsilon^2}{4|\dot{D}(\mu^{\pm}_n)|}\left((b_\varepsilon,\phi^{\pm}_n)_{L_2(Q)} l_\varepsilon\phi^{\pm}_n\right)^2 (1 - 2\varepsilon l_\varepsilon \mathcal{G}^{\pm}_{n,0} b_\varepsilon) + \mathcal{O}(\varepsilon^4).
\end{aligned}$$

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